UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 12 June, 2017 9:00 am to 12:00 pm

PAPER 137

MODULAR FORMS AND L-FUNCTIONS

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

State and prove the Poisson summation formula for $\mathbb{R}$. Use it to obtain the transformation formula for the θ-function

$$\theta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} = t^{-1/2}\theta(1/t), \qquad t > 0.$$

Deduce the analytic continuation and functional equation for the Riemann ζ-function.

[You may assume that $e^{-\pi x^2}$ is its own Fourier transform.]

2

Let $\chi\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character. Define the Dirichlet L-function $L(\chi, s)$. Using a suitable Mellin transform, show that if $\chi \neq \chi_0$, then $L(\chi, s)$ can be analytically continued to the left of the line $\mathrm{Re}(s) = 1$.

Assuming the result that for all $\chi \neq \chi_0$, $L(\chi, 1) \neq 0$, show that the series

$$F(s) = \sum_{\substack{p \text{ prime} \\ (p,N)=1}} \chi(p)p^{-s}\,,$$

for real $s > 1$, satisfies

$$\begin{array}{ll} F(s) \to \infty & \text{as } s \to 1 \text{ if } \chi = \chi_0 \\ F(s) \text{ is bounded} & \text{as } s \to 1 \text{ if } \chi \neq \chi_0. \end{array}$$

Use this to prove Dirichlet's theorem on primes in arithmetic progressions.

[Throughout this question you may use without proof any facts about characters of finite abelian groups you need, as well as the results of Question 1.]

3

Show that there is no nonzero modular form of weight 2 on $\Gamma(1) = SL_2(\mathbb{Z})$. (You may use any result about the number of zeros of a modular form, but should state it clearly.)

Let

$$G_2(z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty}{}' \frac{1}{(mz+n)^2}$$

where Σ' denotes that the term $(m,n) = (0,0)$ is omitted.

a) Show that $G_2(z) = \dfrac{\pi^2}{3} E_2(z)$, where

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n \qquad (q = e^{2\pi i z}).$$

b) Assuming the relation $E_2(-1/z) = z^2 E_2(z) + 12z/2\pi i$, show that if $k > 0$ and $f \in M_k(\Gamma(1))$ then

$$g = \frac{1}{2\pi i} \frac{df}{dz} - \frac{k}{12} E_2 f$$

belongs to $M_{k+2}(\Gamma(1))$, and that g is a cusp form if and only if f is.

c) Let $\Delta = \sum_{n \geqslant 1} \tau(n) q^n$ be the normalised cusp form of weight 12. Show that

$$(1-n)\tau(n) = 24 \sum_{r=1}^{n-1} \sigma_1(r) \tau(n-r).$$

4

Let $\Gamma \subset \Gamma(1) = SL_2(\mathbb{Z})$ be a subgroup of finite index. Show that for any $\gamma \in G = GL_2(\mathbb{Q})^+$, the subgroup $\Gamma' = \Gamma(1) \cap \gamma^{-1}\Gamma\gamma$ has finite index in $\Gamma(1)$.

What is a modular form of weight k on Γ? Explain the condition of holomorphy at the cusps of Γ, and show that if $f \in M_k(\Gamma)$ and $\gamma \in G$ then $f|_k\gamma$ is holomorphic at infinity. Deduce that $f|_k\gamma \in M_k(\Gamma')$.

Let $\chi\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character, where $N > 1$. For $f \in S_k(\Gamma(1))$, define

$$f_\chi(z) = \sum_{\substack{1 \leqslant j \leqslant N \\ (j,N)=1}} \chi(j)^{-1} f\big(z + \frac{j}{N}\big).$$

Show that $f_\chi \in S_k(\Gamma_1(N) \cap \Gamma_0(N^2))$ and that its q-expansion is a constant multiple of

$$\sum_{\substack{n \geqslant 1 \\ (n,N)=1}} \chi(n) a_n(f) q^n.$$

[Here $\Gamma_0(N)$ and $\Gamma_1(N)$ denote the subgroups

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \;\middle|\; c \equiv 0 \pmod{N} \right\}, \quad \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \;\middle|\; c \equiv 0,\ a \equiv d \equiv 1 \pmod{N} \right\}$$

respectively.]

5

Let $k > 0$ be a positive even integer.

a) Show that if $f \in S_k(\Gamma(1))$ then $y^{k/2}\,|f|$ is bounded on the upper half-plane. Use this to prove that $|a_n(f)| < Cn^{k/2}$ for some constant C.

b) Define the Hecke operators $T(n)$ as elements of the algebra of double cosets for $(GL_2(\mathbb{Q})^+, \Gamma(1))$, and show that the associated operators $T_n = n^{k/2-1}T(n)$ acting on $M_k(\Gamma(1))$ are given by

$$T_n f = n^{k/2-1} \sum_{\substack{a,d \geqslant 1 \\ ad=n \\ 0 \leqslant b < d}} f \,\Big|_k \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

(Any results you need about subgroups of $\mathbb{Z}^2$ should be proved.)

Determine the action of T_n on q-expansions of modular forms, and show that if $T_n f = \lambda f$ then $a_n(f) = \lambda a_1(f)$.

Show also that if $a_0(f) \neq 0$ and f is an eigenfunction of all the T_n, then f is a multiple of E_k.

END OF PAPER