UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 2 June, 2017 9:00 am to 12:00 pm

PAPER 136

LOCAL FIELDS

Attempt all ***FOUR*** *questions.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

State a version of Hensel's Lemma.

Let K be a field equipped with a non-archimedean absolute value $|-|$. We denote the valuation ring of K by $\mathcal{O}_K$ and the residue field by k_K. Assume that, for any finite extension L/K, there is a unique extension of $|-|$ to L (also denoted by $|-|$). Let $f \in \mathcal{O}_K[X]$ be a monic irreducible polynomial of degree n, with reduction $\bar{f} \in k_K[X]$. Let M/K be a splitting field of f and let $\alpha_1, \ldots, \alpha_n$ be the roots of f in M (with multiplicities). Show that $|\alpha_i| = |\alpha_1| \leqslant 1$ for all $i = 1, \ldots, n$, and that $\bar{f}(X) = \varphi(X)^m$ for some monic irreducible polynomial $\varphi \in k_K[X]$ and integer $m \in \mathbb{Z}_{\geqslant 1}$.

Deduce that if $g \in \mathcal{O}_K[X]$ is a monic polynomial with reduction $\bar{g} \in k_K[X]$, and there is a factorization $\bar{g} = \bar{g}_1\bar{g}_2$ in $k_K[X]$ with $\bar{g}_1$, $\bar{g}_2$ monic and coprime, then there are monic polynomials $g_1, g_2 \in \mathcal{O}_K[X]$, with reductions $\bar{g}_1$, $\bar{g}_2$ respectively, such that $g = g_1g_2$.

2 Let $\mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p)$ be the space of continuous functions $\mathbb{Z}_p \to \mathbb{Q}_p$. Define the difference operator Δ on $\mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p)$ and the Mahler coefficients of a function $f \in \mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p)$. State Mahler's Theorem and prove it, under the assumption that the Mahler coefficients tend to 0 [You may use standard identities for binomial coefficients without proof. You may use explicit formulae for Δ^n, provided they are clearly stated].

Let $T : \mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p) \to \mathbb{Q}_p$ be a linear function which is translation-invariant (i.e. if $h(x) = f(x+a)$ for some $a \in \mathbb{Z}_p$, then $T(f) = T(h)$). Show that $T = 0$.

3 Define the lower and upper ramification groups of a finite Galois extension L/K of local fields.

Let $K = \mathbb{F}_p((t))$. Let L be the extension of K obtained by adjoining a root of $f(X) = X^p - X - t^{1-p}$. Show that L/K is Galois and compute the lower and upper ramification groups of L/K [Results on computing ramification groups may be used as long as they are clearly stated][*If α and β are roots of f, it might be helpful to consider* $(\beta - \alpha)^p$].

4 Write an essay on Lubin–Tate extensions. You could start by defining formal groups, and focus on (and finish by) sketching the computation of the Galois groups of Lubin–Tate extensions [You do not need to discuss the relation with Local Class Field Theory].

END OF PAPER