UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 8 June, 2017 1:30 pm to 3:30 pm

PAPER 134

LINEAR SYSTEMS

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

In this question all schemes and varieties are assumed to be defined over an algebraically closed field.

(i) State *Nakai's criterion* and explain how that implies that *ampleness* is a numerical condition.

(ii) Let X be a projective scheme and D an effective Cartier divisor. Assume that $\mathcal{O}_D(D)$ is ample.

Show that D is semiample.

(iii) Let X be a projective scheme and let D be a Cartier divisor on X.

Show that the following are equivalent:

(a) D is ample;

(b) for every proper reduced irreducible subvariety $V \subset X$ of positive dimension

$$\chi(V, \mathcal{O}_V(mD)) \to \infty \ \text{ when } m \to \infty;$$

(c) for every proper reduced irreducible subvariety $V \subset X$ of positive dimension there is a positive integer m and a non-zero section $s \in H^0(V, \mathcal{O}_V(mD))$ such that s vanishes at some point of V.

[*Hint: show that (a) is equivalent to (b) and (a) is equivalent to (c).*

For the former equivalence, try to imitate the proof of Nakai's criterion.

For the latter equivalence, use part (ii).]

2

In this question X will always denote a smooth projective surface defined over an algebraically closed field k.

(i) Let D be an effective Cartier divisor on X and C a proper irreducible reduced curve on X. Assume that $D \cdot C < 0$.

Show that $|D| = |D - C| + C$.

(ii) Let D be an effective Cartier divisor on X and C a proper irreducible reduced curve on X. Assume that $D \cdot C < 0$ and $(D - C) \cdot C \leqslant 0$.

Show that $|mD| = |m(D - C)| + mC$.

Use this result to show that if $\psi \colon \widetilde{X} \to X$ is the blowup of X at a smooth point, then $H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}})) = H^0(X, \mathcal{O}_X(mK_X))$, for every $m \geqslant 0$.

(iii) Let X be smooth and projective. Let $C \subset X$ be a proper smooth rational curve. Assume that $C^2 = 0$.

Show that C is a semiample divisor and that the Iitaka dimension of C is 1.

(iv) In this last part, you should assume that K_X is big.

Prove the following assertions.

(a) If K_X is not nef, consider the set

$$\mathrm{Neg}(K_X) := \{ \quad C \subset X \mid \dim C = 1,\ C \text{ is proper, reduced, irreducible and } K_X \cdot C < 0\}.$$

Show that $\mathrm{Neg}(K_X)$ is a finite set.

(b) If K_X is not nef, show that every element of $\mathrm{Neg}(K_X)$ is a smooth rational curve of self-intersection -1.

[*Hint: Show that if* $C \in \mathrm{Neg}(K_X)$ *and* $C^2 \geqslant 0$ *then* $\mathrm{Neg}(K_X)$ *is not a finite set. Use part (iii) for the case* $C^2 = 0$.]

(c) State *Castelnuovo's criterion.*

If K_X is not nef, let C be an element of $\mathrm{Neg}(K_X)$. Show that there exists a smooth surface X_1 and a birational map $f_1 \colon X \to X_1$ which contracts C to a point and is an isomorphism on $X \setminus C$.

3

In this question all schemes and varieties are assumed to be defined over an algebraically closed field.

(i) Define *ampleness* and *bigness* for $\mathbb{R}$-Cartier $\mathbb{R}$-divisors.

(ii) Let X be a projective scheme. Let D a big $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X.

Show that there exists finitely many proper subvarieties of codimension 1, $E_1, \dots, E_k \subset X$ such that if $C \subset X$ is an irreducible reduced proper curve such that $D \cdot C < 0$, then C is contained in one of the E_i.

Assume now that D is ample when restricted to any proper codimension 1 subvariety of X. Let A be an ample divisor on X. Show that there exists $\epsilon > 0$ such that $D - \epsilon A$ is nef on X.

(iii) Let X be a projective scheme. Let D be a class in $\mathrm{N}^1(X)_{\mathbb{R}}$ and assume that

$$D^{\dim V} \cdot [V] > 0,$$

for every irreducible reduced subvariety $V \subset X$ of positive dimension.

(a) Show that D is nef.

(b) Show that there exists ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisors A, A' such that:

- the numerical equivalence classes of $D + A', A + A'$ are ample and rational, and
- $(D + A')^n > n \cdot (D + A')^{n-1} \cdot (A + A')$, where $n = \dim X$.

Conclude that $D - A$ must be big.

(c) Show $D - \epsilon A$ is nef for $0 < \epsilon \ll 1$.

For the purpose of this question, you can assume that you know inductively that D is ample when restricted to any proper subscheme of X.

[*Hint: Use part (ii).*]

(d) Show that D is ample.

END OF PAPER