LINEAR SYSTEMS

Attempt no more than TWO questions.

There are THREE questions in total.

The questions carry equal weight.
In this question all schemes and varieties are assumed to be defined over an algebraically closed field.

(i) State Nakai’s criterion and explain how that implies that ampleness is a numerical condition.

(ii) Let $X$ be a projective scheme and $D$ an effective Cartier divisor. Assume that $\mathcal{O}_D(D)$ is ample. Show that $D$ is semiample.

(iii) Let $X$ be a projective scheme and let $D$ be a Cartier divisor on $X$. Show that the following are equivalent:

(a) $D$ is ample;
(b) for every proper reduced irreducible subvariety $V \subset X$ of positive dimension
   \[ \chi(V, \mathcal{O}_V(mD)) \to \infty \text{ when } m \to \infty; \]
(c) for every proper reduced irreducible subvariety $V \subset X$ of positive dimension there is a positive integer $m$ and a non-zero section $s \in H^0(V, \mathcal{O}_V(mD))$ such that $s$ vanishes at some point of $V$.

[Hint: show that (a) is equivalent to (b) and (a) is equivalent to (c). For the former equivalence, try to imitate the proof of Nakai’s criterion. For the latter equivalence, use part (ii).]
In this question $X$ will always denote a smooth projective surface defined over an algebraically closed field $k$.

(i) Let $D$ be an effective Cartier divisor on $X$ and $C$ a proper irreducible reduced curve on $X$. Assume that $D \cdot C < 0$.
Show that $|D| = |D - C| + C$.

(ii) Let $D$ be an effective Cartier divisor on $X$ and $C$ a proper irreducible reduced curve on $X$. Assume that $D \cdot C < 0$ and $(D - C) \cdot C \leq 0$.
Show that $|mD| = |m(D - C)| + mC$.
Use this result to show that if $\psi: \tilde{X} \to X$ is the blowup of $X$ at a smooth point, then $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(mK_{\tilde{X}})) = H^0(X, \mathcal{O}_X(mK_X))$, for every $m \geq 0$.

(iii) Let $X$ be smooth and projective. Let $C \subset X$ be a proper smooth rational curve. Assume that $C^2 = 0$.
Show that $C$ is a semiample divisor and that the Iitaka dimension of $C$ is 1.

(iv) In this last part, you should assume that $K_X$ is big.
Prove the following assertions.

(a) If $K_X$ is not nef, consider the set
$$\text{Neg}(K_X) := \{ C \subset X \mid \dim C = 1, C \text{ is proper, reduced, irreducible and } K_X \cdot C < 0\}.$$ 
Show that $\text{Neg}(K_X)$ is a finite set.

(b) If $K_X$ is not nef, show that every element of $\text{Neg}(K_X)$ is a smooth rational curve of self-intersection $-1$.
[Hint: Show that if $C \in \text{Neg}(K_X)$ and $C^2 \geq 0$ then $\text{Neg}(K_X)$ is not a finite set. Use part (iii) for the case $C^2 = 0$.]

(c) State Castelnuovo's criterion.
If $K_X$ is not nef, let $C$ be an element of $\text{Neg}(K_X)$. Show that there exists a smooth surface $X_1$ and a birational map $f_1: X \to X_1$ which contracts $C$ to a point and is an isomorphism on $X \setminus C$. 

Part III, Paper 134

[TURN OVER
In this question all schemes and varieties are assumed to be defined over an algebraically closed field.

(i) Define *ampleness* and *bigness* for \( \mathbb{R} \)-Cartier \( \mathbb{R} \)-divisors.

(ii) Let \( X \) be a projective scheme. Let \( D \) a big \( \mathbb{R} \)-Cartier \( \mathbb{R} \)-divisor on \( X \).

Show that there exists finitely many proper subvarieties of codimension 1, \( E_1, \ldots, E_k \subset X \) such that if \( C \subset X \) is an irreducible reduced proper curve such that \( D \cdot C < 0 \), then \( C \) is contained in one of the \( E_i \).

Assume now that \( D \) is ample when restricted to any proper codimension 1 subvariety of \( X \). Let \( A \) be an ample divisor on \( X \). Show that there exists \( \epsilon > 0 \) such that \( D - \epsilon A \) is nef on \( X \).

(iii) Let \( X \) be a projective scheme. Let \( D \) be a class in \( N^1(X)_\mathbb{R} \) and assume that

\[
D^{\dim V} \cdot [V] > 0,
\]

for every irreducible reduced subvariety \( V \subset X \) of positive dimension.

(a) Show that \( D \) is nef.

(b) Show that there exists ample \( \mathbb{R} \)-Cartier \( \mathbb{R} \)-divisors \( A, A' \) such that:

- the numerical equivalence classes of \( D + A', A + A' \) are ample and rational, and
- \( (D + A')^n > n \cdot (D + A')^{n-1} \cdot (A + A') \), where \( n = \dim X \).

Conclude that \( D - A \) must be big.

(c) Show \( D - \epsilon A \) is nef for \( 0 < \epsilon \ll 1 \).

For the purpose of this question, you can assume that you know inductively that \( D \) is ample when restricted to any proper subscheme of \( X \).

[Hint: Use part (ii).]

(d) Show that \( D \) is ample.

END OF PAPER