UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 8 June, 2017 9:00 am to 11:00 am

PAPER 133

GEOMETRIC GROUP THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) State the Milnor-Svarc theorem.

(b) Let G act by isometries on the geodesic metric space X (*the action* ***need not be*** *properly discontinuous*). Suppose that the action is cobounded (*i.e. there exists a ball* $B \subset X$ *with* $\cup_{g \in G} gB = X$). State a weaker version of the Milnor-Svarc theorem that holds in this situation.

(c) Explain what modifications need to be made to the proof of the Milnor-Svarc theorem in order to obtain a proof of your statement from part (b). (*You do* ***not*** *need to write a complete proof.*)

(d) Let $A = \langle a, a' \mid aa'a^{-1}(a')^{-1} \rangle$ and let $B = \langle b, b' \mid bb'b^{-1}(b')^{-1} \rangle$ and consider the free product

$$A * B \cong \langle a, a', b, b' \mid aa'a^{-1}(a')^{-1},\ bb'b^{-1}(b')^{-1} \rangle.$$

Let $\mathcal{T}$ be the graph with a vertex for each coset of A in $A * B$, and a vertex for each coset of B in $A * B$, with vertices gA and hB adjacent in $\mathcal{T}$ if and only if $gA \cap hB \neq \emptyset$. You may assume without proof that $\mathcal{T}$ is a tree.

Find an infinite generating set of $A * B$ so that the corresponding *(locally infinite)* Cayley graph of $A * B$ is quasi-isometric to $\mathcal{T}$.

Explain why no Cayley graph coming from a finite generating set of $A * B$ is quasi-isometric to a tree.

2

*(**Note:** In this question, given a 2–complex A, we denote by $A^{(1)}$ its 1–skeleton, which we endow with the usual path-metric in which each edge has length 1.)*

(a) Let M be a geodesic metric space. Say what it means for $N \subset M$ to be C–*quasiconvex*, where $C \in \mathbb{R}$.

Let X be a compact 2–complex satisfying the $C'(1/6)$ small-cancellation condition.

(b) State the Greendlinger theorem for X.

Let $\widetilde{X}$ be the universal cover of X. Let $\widetilde{Y} \subset \widetilde{X}$ be a connected subcomplex with the following property:

for any 2–cell R of $\widetilde{X}$, either R lies in $\widetilde{Y}$ or $\partial_p R = OI$, where $|I| \geqslant |O|$ and no 1–cell traversed by I lies in $\widetilde{Y}$. $(\star)$

(c) Let $p, q \in \widetilde{Y}$ be 0–cells and let γ be a geodesic of $\widetilde{X}^{(1)}$ joining p to q. Let σ be a combinatorial geodesic of $\widetilde{Y}^{(1)}$ joining p to q. Say why there is a reduced disc diagram $D_\sigma \to \widetilde{X}$ with boundary path $\gamma\sigma^{-1}$.

(d) By considering σ and D_σ as above, prove that $\widetilde{Y}^{(1)}$ is a quasiconvex subspace of $\widetilde{X}^{(1)}$.

3

(a) Let G be a hyperbolic group. Let $H \cong \langle a, b \mid aba^{-1} = b^3 \rangle$. Show that there is no injective homomorphism $H \to G$.

(b) Let

$$H = \langle a, b, c \mid aba^{-1}b^{-1} = c,\ bcb^{-1}c^{-1} = cac^{-1}a^{-1} = 1 \rangle.$$

Prove that the inclusion $\langle c \rangle \to H$ is not a quasi-isometric embedding, where $\langle c \rangle \cong \mathbb{Z}$ is endowed with the metric coming from any finite generating set. (*Hint: first prove that $a^k b^k a^{-k} b^{-k} = c^{k^2}$ for all $k \geqslant 0$.*)

4

(a) Give the definition of a *Dehn presentation.*

(b) Let $\langle A \mid R\rangle$ be a Dehn presentation for a group G. Suppose $g \in G$ has finite order $n > 1$. Let w be a word over the alphabet $A \cup A^{-1}$ so that w represents an element of G which is conjugate to g, and w is of minimal length with this property. Prove that there exists $r \in R$ so that the word w^n has a subword u so that u is a subword of r and $|u| > |r|/2$.

(c) Deduce that $|w| \leqslant |r|/2 + 2$.

(d) By considering $M = \max\{|r|/2 + 2 \mid r \in R\}$, prove that G has only finitely many conjugacy classes of finite-order elements.

END OF PAPER