UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 7 June, 2017 9:00 am to 12:00 pm

PAPER 132

RIEMANN SURFACES AND TEICHMÜLLER THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let $\mathcal{U} \subset \mathbb{C}$ be a non-empty simply-connected open proper subset. Prove that there exists a holomorphic bijection $f : \mathcal{U} \to \mathbb{H}$, where $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.

(b) Let

$$\Gamma(2) = \{A \in \mathrm{SL}(2, \mathbb{R}) : A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod (2)\},$$

acting on $\mathbb{H}$ by Möbius transformations. Prove that $\mathbb{H}/\Gamma(2) \cong \mathbb{C} \setminus \{0, 1\}$.

(c) Show that $\mathbb{C} \setminus \{0, 1\}$ is equipped with a complete conformal metric of constant negative curvature and compute the length of its shortest closed geodesic.

2

(a) Explain, carefully, why there is a natural action of the group $\mathrm{SL}(2, \mathbb{R})$ on the moduli space of holomorphic 1-forms $\Omega\mathcal{M}_g = \{(X, \omega) : \omega \in \mathrm{H}^0(X, K_X)\}$ and holomorphic quadratic differentials $\mathcal{QM}_g = \{(X, q) : q \in \mathrm{H}^0(X, K_X^2)\}$, respectively. Prove that the map $s((X, \omega)) = (X, \omega^2)$, $s : \Omega\mathcal{M}_g \to \mathcal{QM}_g$ is $\mathrm{SL}(2, \mathbb{R})$-equivariant.

(b) Is there a similar natural action of $\mathrm{SL}(2, \mathbb{R})$ on the moduli space of holomorphic cubic differentials $\mathcal{CM}_g = \{(X, c) : c \in \mathrm{H}^0(X, K_X^3)\}$?

(c) Let $n \in \mathbb{N}$, with $n > 1$, and (X_n, ω_n) be the 1-form obtained by identifying the opposite sides of a regular euclidean $2n$-gon, of area 1, by translation. Prove, carefully, that X_n is a closed Riemann surface and compute its genus. Do the 1-forms (X_4, ω_4) and (X_5, ω_5) belong to the same $\mathrm{SL}(2, \mathbb{R})$-orbit?

3

(a) Let Γ be a path family on a Riemann surface X. Define the *extremal length* $\lambda(\Gamma, X)$ of Γ.

(b) Let $f : X \to Y$ be a smooth orientation preserving diffeomorphism between two Riemann surfaces X, Y, which is K-quasiconformal, for some $K > 1$. Prove that $\lambda(f(\Gamma), Y) \leqslant K \cdot \lambda(\Gamma, X)$.

(c) State and prove Teichmüller's uniqueness theorem.

(any results quoted from lectures should be clearly stated.)

4

(a) State and prove Mumford's compactness theorem.

(b) Let $(X_n, q_n) \in \mathcal{QT}_g$, $n = 1, 2, \ldots$ $(g \geqslant 2)$ be a sequence of area one quadratic differentials and γ a simple closed curve with $\ell_{X_n}(\gamma) \to 0$, as $n \to \infty$, in the hyperbolic metric of X_n, Prove that $L(\gamma, |q_n|^{1/2}) \to 0$ as $n \to \infty$, where $L(\gamma, |q_n|^{1/2}) = \inf\{\int_{\widetilde{\gamma}} |q_n|^{1/2}\}$ and the infimum is over all closed curves $\widetilde{\gamma}$ homotopic to γ. Is the converse true?

END OF PAPER