UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 12 June, 2017 1:30 pm to 4:30 pm

PAPER 131

RIEMANNIAN GEOMETRY

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) What is the *curvature 2-form* R of a Riemannian manifold (M, g)? Prove the first Bianchi identity $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$, where X, Y, Z are vector fields on M.

Define the *Ricci curvature* Ric and the *sectional curvature* K. Show that if $\dim M = 3$ then the components $K(e_i, e_j)$, for an orthonormal basis $\{e_1, e_2, e_3\}$, are determined by the Ricci curvature. [Algebraic symmetries of the Riemann curvature may be assumed.]

(b) Let $f : M \to N$ be a local isometry between Riemannian manifolds, where N is complete. Show that M is complete if and only if f is a covering map.

[*The Hopf–Rinow theorem may be assumed if accurately stated.*]

2

Define the *exponential map* $\exp_p$ and *geodesic coordinates* for a point p in a Riemannian manifold. Show the geodesic coordinates are well-defined on some open neighbourhood of p. What is a *geodesic sphere*? State the Gauss lemma.

Define the distance function $d(\cdot,\cdot)$ induced by a metric on a connected Riemannian manifold M. Given a point $p \in M$, prove that for each $q \in M$, $q \neq p$, and each sufficiently small $\delta = \delta(q) > 0$ there exists $p_0 \in M$ such that $d(p, p_0) = \delta$ and $d(p, p_0) + d(p_0, q) = d(p, q)$.

Let G be a Lie group and ∇ the Levi-Civita connection of a bi-invariant (i.e. left- and right-invariant) Riemannian metric on G. Prove that $\nabla_X X = 0$ for every left-invariant vector field X on G. Deduce that every geodesic $\gamma(t)$ on G starting at the identity element $\gamma(0) = I \in G$ is defined by a group homomorphism $\mathbb{R} \to G$.

[*Standard existence and uniqueness results for integral curves of vector fields may be assumed if accurately stated.*]

3

Let $\gamma(t)$, $0 \leqslant t \leqslant 1$ be a geodesic curve in an n-dimensional Riemannian manifold ($n \geqslant 2$). Define the *Jacobi fields* along γ. What is a *geodesic variation* of γ? Prove that every Jacobi field along γ arises from a geodesic variation.

Explain why the Jacobi fields J along γ point-wise normal to γ and such that $J(0) = 0$, $J(1) = 0$ form a vector space of dimension not greater than $n - 1$. Give an example when the dimension $n - 1$ is attained.

Suppose now that M is an orientable Riemannian manifold of even dimension with positive sectional curvature and γ_0 is a closed geodesic in M. Prove that γ_0 is homotopic to a closed curve with length strictly smaller than that of γ_0.

[*You may assume the formula for the second variation of energy. Algebraic symmetries of the Riemann curvature tensor may be assumed.*]

4

Explain how a Riemannian metric on a manifold induces an inner product on the fibres of the bundle of differential forms of every given degree. Define the *Hodge $*$-operator* and the *Laplace–Beltrami operator* Δ. Prove that a differential form α on M is harmonic if and only if $*\alpha$ is so.

State the Hodge decomposition theorem.

Let M be a compact oriented Riemannian manifold and f a smooth function on M. Prove that if $\Delta f \geqslant 0$ then f is constant on M. Prove that every de Rham cohomology class on M is represented by a unique harmonic form.

Suppose that M also has a smooth right action of a connected Lie group G, i.e. a smooth map $R : M \times G \to M$, such that $R(x, e) = x$ for the identity element $e \in G$ and $R(R(x, h_1), h_2) = R(x, h_1 h_2)$, for all $x \in M$ and $h_1, h_2, \in G$. Suppose further that for every $h \in G$, the map $R_h = R(\,\cdot\,, h)$ is an isometry of M. Show that every harmonic form α on M is invariant under the action of G, i.e. $R_h^* \alpha = \alpha$ for all h.

[*You may assume that the operator $\delta = (-1)^{n(p+1)+1} * d *$ on p-forms ($n = \dim M$, $p > 0$) is the formal adjoint of the exterior derivative d. Standard results about de Rham cohomology and smooth homotopy may be assumed if accurately stated.*]

5

State the Bochner–Weitzenböck formula for 1-forms. [You should include careful definitions of the terms that appear in the formula.]

Define the *holonomy group* of a connected Riemannian manifold (M, g). What is the holonomy representation? State the fundamental principle of Riemannian holonomy.

Suppose that (M, g) is a compact Riemannian manifold with $\mathrm{Ric}(g) \geqslant 0$ at each point and with irreducible holonomy representation. Suppose further that M admits a covering $X \times T^k \to M$, for some $0 \leqslant k \leqslant \dim M$, where X is a compact simply-connected manifold and T^k is a k-dimensional torus. Show that the fundamental group of M is finite.

[*You may assume that every cohomology class on a compact Riemannian manifold is represented by a harmonic form. You may also assume M and its finite cover have the same Betti numbers.*]

END OF PAPER