UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 8 June, 2017 1:30 pm to 3:30 pm

PAPER 130

RAMSEY THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Prove Ramsey's Theorem for graphs, i.e., show that whenever $\mathbb{N}^{(2)}$ is finitely coloured, there exists an infinite set $X \subset \mathbb{N}$ such that $X^{(2)}$ is monochromatic.

(ii) Deduce from Ramsey's Theorem that whenever $\mathbb{N}$ is finitely coloured, there exists an infinite sequence $x_1 < x_2 < x_3 < \dots$ of natural numbers such that the set $\{x_i x_j^2 : i < j\}$ is monochromatic.

(iii) Show that it is possible to finitely colour the *real numbers* in the interval $[1, \infty)$ in such a way that for each $x \in [1, \infty)$, the colour of x differs from that of each real number in the interval $[1.9x, 2x]$.

(iv) Using part (iii) or otherwise, show that it is *NOT TRUE* that whenever $\mathbb{N}$ is finitely coloured, there exists an infinite sequence $x_1 < x_2 < x_3 < \dots$ of natural numbers such that the set $\{x_i x_j^2 : i \neq j\}$ is monochromatic.

2

(i) Prove the Hales–Jewett Theorem, i.e., show that for any $m, k \in \mathbb{N}$, there exists an $n = n(m, k) \in \mathbb{N}$ such that whenever $[m]^n$ is k-coloured, there exists a monochromatic combinatorial line.

(ii) Deduce van der Waerden's Theorem from the Hales–Jewett Theorem.

(iii) A subset of $[N]$ is called an *interval* if it is of the form $\{a, a+1, a+2, \dots, b-1, b\}$ for some $1 \leqslant a \leqslant b \leqslant N$.

Show that the following strengthening of the Hales–Jewett Theorem is *NOT TRUE*: for any $m, k \in \mathbb{N}$, there exists an $N = N(m, k) \in \mathbb{N}$ such that whenever $[m]^N$ is k-coloured, there exists a monochromatic combinatorial line whose active coordinate set is an interval.

[*Hint: Consider, for example, the line* $\{(2, 2, 5, x, x, x, 7, 9) : 1 \leqslant x \leqslant 9\}$ *in* $[9]^8$*; what distinguishes the two points corresponding to* $x = 5$ *and* $x = 7$ *from the seven other points on this line?*]

3

(i) Show that if $a_1, \ldots, a_n$ are non-zero rationals, then the matrix $(a_1, \ldots, a_n)$ is partition-regular if and only if some (non-empty) subset of the a_i sum to zero.

[You may assume van der Waerden's Theorem. No form of Rado's Theorem may be assumed without proof.]

(ii) Suppose that $a_1, \ldots, a_n$ are non-zero rationals such that the matrix $(a_1, \ldots, a_n)$ is partition-regular. Show that whenever $\mathbb{N}$ is finitely coloured, there exists a monochromatic solution $(y_1, \ldots, y_n) \in \mathbb{N}^n$ to the equation

$$\frac{a_1}{y_1} + \cdots + \frac{a_n}{y_n} = 0.$$

[*Hint: Suppose that T is a natural number with the property that there exists a monochromatic solution to $a_1x_1 + \cdots + a_nx_n = 0$ in $[T]$ whenever $[T]$ is k-coloured. What can you now say about a k-colouring of $[S]$ where $S = \operatorname{lcm}\{1, 2, \ldots, T\}$?*]

4

(i) Consider the dynamical system $(\mathcal{C}, \mathcal{L})$, where $\mathcal{C} = \{c : \mathbb{Z} \to [k]\} = [k]^{\mathbb{Z}}$ is the space of all k-colourings of $\mathbb{Z}$, and $\mathcal{L}$ denotes the left-shift operator on this space.

Show that a colouring $c \in \mathcal{C}$ is minimal if and only if c has the 'bounded gaps property', i.e., if for each interval $I \subset \mathbb{Z}$, there exists an M such that $c(I)$ appears as a contiguous subsequence of $c(U)$ for each interval $U \subset \mathbb{Z}$ of length at least M.

(ii) State and prove Hindman's Theorem.

[You may assume the following result: if (X, T) is a dynamical system such that $X = \bar{x}$ for some $x \in X$, and $Y \subset X$ is minimal, then there exists a $y \in Y$ such that x and y are proximal.]

(iii) Does there exist a left-shift invariant metric d on $[2]^{\mathbb{Z}}$, i.e., a metric d such that

$$d(x, y) = d(\mathcal{L}(x), \mathcal{L}(y))$$

for all $x, y \in [2]^{\mathbb{Z}}$, which induces the (usual) product topology on $[2]^{\mathbb{Z}}$?

[*Hint: Can you find two distinct points $x, y \in [2]^{\mathbb{Z}}$ such that both $\mathcal{L}^n(x)$ and $\mathcal{L}^n(y)$ converge to the same point as $n \to \infty$?*]

END OF PAPER