UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 1 June, 2017 9:00 am to 11:00 pm

PAPER 129

INTRODUCTION TO ADDITIVE COMBINATORICS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $\delta > 0$, let n be an odd positive integer and let A be a subset of $\{1, 2, \ldots, n\}$ of cardinality at least δn. Prove that either A contains an arithmetic progression of length 3 or there is an arithmetic progression $P \subset \{1, 2, \ldots, n\}$ of length at least $c_1\sqrt{n}$ such that $|A \cap P| \geqslant \delta(1 + c_2\delta)|P|$, where c_1 depends on δ only, c_2 is an absolute constant, and c_1 and c_2 are both positive.

2

State and prove Szemerédi's regularity lemma.

The *half graph* of size n is the bipartite graph with two vertex sets X and Y that are both copies of $\{1, 2, \ldots, n\}$, with $x \in X$ joined to $y \in Y$ if and only if $x \leqslant y$. Suppose that X and Y are partitioned into sets X_i and Y_j respectively, with each X_i and each Y_j of size at least 100.

For each positive integer m with $1 \leqslant m \leqslant n$ there exist unique i and j such that $m \in X_i$ and $m \in Y_j$. Call m *close to its X-boundary* if fewer than $|X_i|/10$ points in X_i are greater than m or fewer than $|X_i|/10$ points in X_i are less than m. Define closeness to the Y boundary in a similar way.

Obtain an upper bound for the number of elements of $\{1, 2, \ldots, n\}$ that are close to their X or Y boundaries, and deduce that not every pair (X_i, Y_j) is $(1/10)$-regular. (The constant $1/10$ is not optimized.)

3

Prove that there exists a constant $C < 3$ such that if A is any subset of $\mathbb{F}_3^n$ of size at least C^n, then there are distinct elements $x, y, z \in A$ such that $x + y + z = 0$. [You may assume standard probabilistic estimates, provided that you state them carefully.]

4

Let X and Y be finite sets and let $f : X \times Y \to \mathbb{R}$. Define the *box norm* $\|f\|_\square$ of f and prove that $\|.\|_\square$ is a norm. Prove also that if $u : X \to \mathbb{R}$ and $v : Y \to \mathbb{R}$, then

$$|\mathbb{E}_{x,y} f(x,y)u(x)v(y)| \leqslant \|f\|_\square \|u\|_2 \|v\|_2.$$

Let G be a tripartite graph with finite vertex sets X, Y and Z. Write $G(X,Y)$ for the bipartite subgraph induced by X and Y, and similarly for the other two pairs. Let α, β and γ be the densities of $G(X,Y)$, $G(Y,Z)$ and $G(X,Z)$, respectively.

Suppose that the function $G(X,Y) - \alpha$ has box norm at most c, and that every vertex in Z is adjacent to exactly $\beta|Y|$ vertices in Y. Prove that the number of triangles in G differs from $\alpha\beta\gamma|X||Y||Z|$ by at most $3c|X||Y||Z|$. [Hint: Let $f_1 = G(X,Y) - \alpha$, $f_2 = G(Y,Z) - \beta$ and $f_3 = G(X,Z) - \gamma$, and consider the expression

$$\mathbb{E}_{x,y,z}(\alpha + f_1(x,y))(\beta + f_2(y,z))(\gamma + f_3(x,z)).$$

END OF PAPER