UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Friday, 9 June, 2017 1:30 pm to 4:30 pm

## PAPER 128

## ALGEBRAS

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

*The questions carry equal weight.*

***Throughout this paper $k$ is a field.***

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

Let $A$ be a $k$-algebra. Define the Jaconson radical $J(A)$.

What does it mean for a finitely generated right $A$-module $P$ to be an indecomposable projective?

Suppose that $A$ is right Artinian. Let $P$ be an an indecomposable projective right submodule of $A$. Show that $P/PJ(A)$ is a simple $A$-module.

Define what is meant by a block of $A$.

Suppose $k$ is algebraically closed and has characteristic 3. Express $kS_3$ as a direct sum of blocks, and as a direct sum of indecomposable projectives.

**2**

Define what it means for a $k$-algebra $A$ to be right Noetherian?

Let $B$ be a an algebra generated by a subalgebra $A$ and an element $x$. Suppose that $A$ is right Noetherian and that $A + xA = A + Ax$. Show that $B$ is right Noetherian.

Deduce that the quantum torus $k_q[X, X^{-1}, Y, Y^{-1}]$ is right Noetherian.

Define what it means for a proper ideal $P$ of $A$ to be prime.

Show that the intersection $N$ of the prime ideals of $A$ is nilpotent, and is the intersection of finitely many prime ideals.

**3**

Show that the quantum plane $k_q[X, Y]$ is a domain.

Let $A$ be a right Noetherian k-algebra which is a domain. What does it mean for a non-zero $A$-module to be uniform. Show that $A$ is a uniform right $A$-module, and that it embeds in a division $k$-algebra.

**4**

Define what is meant by the Gelfand-Kirillov (GK) dimension of a finitely generated $k$-algebra $A$, and also define the GK-dimension of a non-zero finitely generated $A$-module.

Let $S$ be a commutative graded $k$-algebra generated by finitely many elements of degree 1. Show that the GK-dimension of $S$ is an integer.

Let $A$ be the first Weyl algebra $A_1(k)$ where $k$ has characteristic zero. What are the possible GK-dimensions of non-zero finitely generated $A$-modules? Briefly justify your answer giving examples in each case.

**5**

Let $A$ be a $k$-algebra and let $M$ be a finitely generated $A-A$ bimodule.

Define the Hochschild cohomology $HH^n(A,M)$ and also define the (Hochschild cohomological) dimension $\mathrm{Dim}(A)$ of $A$.

What is meant by an extension of $A$ by $M$? Show that there is a one-one correspondence between $HH^2(A,M)$ and the isomorphism classes of extensions of $A$ by $M$.

Define what is meant by a star product on $A \otimes k[[t]]$. What does it mean for a star produce to be trivial?

Show that if $\mathrm{Dim}(A) \leqslant 1$ then any star product on $A \otimes k[[t]]$ is trivial.

**6** Let $k$ be an algebraically closed field of characteristic zero. Let $A$ be a finitely generated commutative $k$ algebra. Define what is meant by a $k$ derivation of $A$. Denote the set of derivations of $A$ by $Der(A)$. Show that $Der(A)$ forms a Lie algebra.

Show that $HH^0(A,A) = A$ and that $HH^1(A,A) = Der(A)$.

Define the cup product and the Gerstenhaber bracket on the Hochschild cochain complex of $A$.

What is meant by a Gerstenhaber algebra?

Calculate the Hochschild cohomology of $k[X]$ where $k$ is algebraically closed of characteristic zero. What is the Gerstenhaber algebra structure?

What is the Gerstenhaber algebra structure on the Hochschild cohomology of $k[X,Y]$? (You may state the HKR theorem for $k[X,Y]$).

## END OF PAPER