UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 8 June, 2017 9:00 am to 12:00 pm

PAPER 121

TOPICS IN SET THEORY

*Attempt all **FOUR** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

In this question, IC stands for the axiom "there is an inaccessible cardinal", i.e., a regular cardinal κ such that for all $\lambda < \kappa$, we have $|\wp(\lambda)| < \kappa$.

(i) Suppose $M \subseteq N$ are classes and $\varphi(v_1, \ldots, v_n)$ is a formula of the language of set theory $\mathcal{L}_\in$. Define the phrase "the formula φ is absolute between M and N".

(ii) Define the class of Δ_0^{ZF}-formulas.

(iii) Define the levels $\mathbf{V}_\alpha$ of the von Neumann hierarchy by transfinite recursion.

(iv) Prove that Δ_0^{ZF}-formulas are absolute between transitive classes $M \subseteq N$ that are models of ZF.

(v) Show that the formula $\varphi(x)$ saying "x is a cardinal" is absolute between transitive models of the form $\mathbf{V}_\alpha$ for limit ordinals α.

(vi) An cardinal α is called *worldly* if $\mathbf{V}_\alpha \models \mathsf{ZFC}$. Work in ZFC and show that if κ is an inaccessible cardinal, then the set of worldly cardinals below κ has size κ.

2

(i) Define the notion of a hierarchy and state the Reflection Theorem for hierarchies.

(ii) Suppose X is any transitive set. Define by transfinite recursion the constructive universe $\mathbf{L}(X)$ over X.

(iii) Prove that there is a formula $\Phi(x)$ with one free variable that has the property that for any transitive sets X and M with $X \in M$, we have that $(M, \in) \models \Phi(X)$ if and only if $M = \mathbf{L}_\alpha(X)$ for a limit ordinal α.

(iv) Suppose that X is a transitive set. Formulate a version of the Condensation Lemma for $\mathbf{L}(X)$ and prove it.

(v) Suppose $\beth_1 = \aleph_2$ and let $X := \wp(\mathbb{N})$. Prove that $\mathbf{L}(X) \models \neg\mathsf{CH}$.

3

In this question, assume that M is a countable transitive model of ZFC, $(\mathbb{P}, \leqslant_{\mathbb{P}}) \in M$ and $(\mathbb{Q}, \leqslant_{\mathbb{Q}}) \in M$ are partial orders, and that G is $\mathbb{P}$-generic over M and H is

(i) Give precise definitions of both the semantic forcing relation $\Vdash$ and the syntactic forcing relation $\Vdash^*$.

(ii) Assume that you already proved for any two $\mathbb{P}$-names σ and τ that $\mathrm{val}(\sigma, G) = \mathrm{val}(\tau, G)$ if and only if there is a $p \in G$ such that $p \Vdash^* \sigma = \tau$. Under this assumption, show that the following are equivalent:

(a) $\mathrm{val}(\sigma, G) \in \mathrm{val}(\tau, G)$ and

(b) there is a $p \in G$ such that $p \Vdash^* \sigma \in \tau$.

(iii) Show that $M[G] \models \mathsf{PowerSet}$.

(iv) Define the *product order* on $\mathbb{P} \times \mathbb{Q}$ by $(p, q) \leqslant (p', q')$ if and only if $p \leqslant_{\mathbb{P}} p'$ and $q \leqslant_{\mathbb{Q}} q'$. Assume that H is $\mathbb{P} \times \mathbb{Q}$-generic over M. Show that there are $G_0 \subseteq \mathbb{P}$ and $G_1 \subseteq \mathbb{Q}$ such that

(a) $H = G_0 \times G_1$,

(b) G_0 is $\mathbb{P}$-generic over M, and

(c) G_1 is $\mathbb{Q}$-generic over $M[G_0]$.

4

In this question, assume that M is a countable transitive model of ZFC and $(\mathbb{P}, \leqslant, \mathbf{1}) \in M$ is a partial order.

(i) Define what an atom of $\mathbb{P}$ is and what it means for $\mathbb{P}$ to be non-atomic.

(ii) Define the partial order $\mathrm{Fn}(I, J)$.

(iii) Show that if G is $\mathbb{P}$-generic over M and $\mathbb{P}$ is non-atomic, then $G \notin M$.

(iv) Show that if G is $\mathrm{Fn}(\aleph_2^M \times \aleph_0, 2)$-generic over M, then in $M[G]$ there is an injection from $\aleph_2^M$ into $\wp(\mathbb{N})$.

(v) Assume that $\mathbb{P}$ is non-atomic. Build the following sequence of models by recursion: $M_0 := M$; if M_i is constructed, find G_i which is $\mathbb{P}$-generic over M_i and let $M_{i+1} := M_i[G_i]$. Let $N := \bigcup_{i \in \omega} M_i$. Prove that N is not a model of $\mathsf{PowerSet}$.

END OF PAPER