UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 1 June, 2017 1:30 pm to 4:30 pm

PAPER 119

CATEGORY THEORY

Attempt no more than ***FOUR*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Explain what is meant by an *equivalence of categories*. Assuming the axiom of choice, show that a functor $F : \mathcal{C} \to \mathcal{D}$ is part of an equivalence if and only if it is full, faithful and essentially surjective on objects. Hence or otherwise show that the category **Part** of sets and partial functions is equivalent to the category $\mathbf{Set}_*$ of pointed sets and basepoint-preserving functions. Are these two categories isomorphic? Justify your answer.

Define the notion of *skeletal category*. Show that if a functor between skeletal categories is part of an equivalence then it is an isomorphism. Show also that the assertion 'Every small category is equivalent to a skeletal category' is equivalent to the axiom of choice. [*Hint: given a family* $(A_i \mid i \in I)$ *of nonempty sets, consider a suitable category whose objects are the members of the disjoint union* $\amalg_{i \in I} A_i$.]

2 What is meant by saying that a category is *balanced*? If $F : \mathcal{C} \to \mathcal{D}$ is a faithful functor and $\mathcal{C}$ is balanced, prove that F reflects isomorphisms.

Let $((F : \mathcal{C} \to \mathcal{D}) \dashv (G : \mathcal{D} \to \mathcal{C}))$ be an adjunction with unit η and counit ϵ. Show that F is faithful if and only if η is a (pointwise) monomorphism. Now suppose that $\mathcal{C}$ is balanced, and that every morphism of $\mathcal{D}$ can be factored as a strong epimorphism followed by a monomorphism. Show that the following are equivalent:

(i) Both η and ϵ are monomorphisms.

(ii) F is full and faithful, and its image is closed under strong quotients in $\mathcal{D}$ (that is, if $FA \to B$ is a strong epimorphism, then B is isomorphic to FA' for some A').

Give an example of an adjunction whose unit and counit are both monic, but whose left adjoint is not full.

3 Define the terms *diagram*, *cone* over a diagram and *limit* of a diagram. Show that if a category has finite products and equalizers then it has all finite limits.

A functor $F : I \to J$ between small categories is called *initial* if, for every object j of J, the category $(F \downarrow j)$ is (nonempty and) connected. If F is initial, show that for any diagram $D\colon J \to \mathcal{C}$ the functor which sends $(\gamma_j \mid j \in \text{ob } J)$ to $(\gamma_{Fi} \mid i \in \text{ob } I)$ is an isomorphism from the category of cones over D to that of cones over DF. Deduce that if $\mathcal{C}$ has limits of shape I then it also has limits of shape J, and the diagram

$$\begin{tikzcd} & {[I, \mathcal{C}]} \arrow[d, "\lim_I"] \\ {[J, \mathcal{C}]} \arrow[ur, "F^*"] \arrow[r, "\lim_J"] & \mathcal{C} \end{tikzcd}$$

commutes up to isomorphism, where F^* denotes the functor $D \mapsto DF$.

Conversely, if this diagram commutes for $\mathcal{C} = \mathbf{Set}^{\text{op}}$, show that F is initial.

[*Consider functors of the form* $J(-, j)$.]

4 Explain briefly what is meant by a *monad*, and by an *algebra* for a monad.

Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on a category $\mathcal{C}$ with finite coproducts, and let F denote the free $\mathbb{T}$-algebra functor. If (A, α) and (B, β) are two $\mathbb{T}$-algebras, show that if the parallel pair

$$F(TA + TB) \underset{\mu_{A+B}.F\kappa}{\overset{F(\alpha+\beta)}{\rightrightarrows}} F(A + B)$$

has a coequalizer in $\mathcal{C}^{\mathbb{T}}$ (where $\kappa\colon TA+TB \to T(A+B)$ is the comparison map defined by $\kappa\nu_i = T\nu_i$ for $i = 1, 2$), then (the codomain of) the coequalizer is a coproduct $(A, \alpha)+(B, \beta)$ in $\mathcal{C}^{\mathbb{T}}$. Deduce that if $\mathcal{C}$ has all finite colimits and T preserves reflexive coequalizers, then $\mathcal{C}^{\mathbb{T}}$ has all finite colimits. [*You may assume the result that the forgetful functor* $\mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ *creates any colimits which are preserved by* T.]

5 Define the notion of *regular category.*

Let $\mathcal{C}$ be a category with pullbacks and images. Show that, for each $f : A \to B$ in $\mathcal{C}$, the functor $f^* : \mathrm{Sub}_{\mathcal{C}}(B) \to \mathrm{Sub}_{\mathcal{C}}(A)$ obtained by pulling back subobjects along f has a left adjoint $\exists_f$, and show that the 'Frobenius reciprocity' condition

$$\exists_f(A' \cap f^*(B')) \cong \exists_f(A') \cap B'$$

holds (for arbitrary subobjects $A' \rightarrowtail A$ and $B' \rightarrowtail B$) if and only if strong epimorphisms in $\mathcal{C}$ are stable under pullback along monomorphisms.

Let $\mathcal{D}$ and $\mathcal{E}$ be isomorphic copies of the category **Rng** of rings (with 1) and let $\mathcal{C}$ be obtained from the disjoint union of $\mathcal{D}$ and $\mathcal{E}$ by identifying their terminal objects and then adjoining a strict initial object. Show that $\mathcal{C}$ has finite limits and images and satisfies Frobenius reciprocity, but is not regular. [*You may assume without proof that* **Rng** *is regular, but you should indicate any other properties of this category which are used in the argument.*]

6 Explain what is meant by a *pointed category* and a *semi-additive category.*

Let $\mathcal{C}$ be a pointed category with finite products and coproducts, in which the canonical morphism $c \colon A + B \to A \times B$ with matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is an isomorphism for each pair of objects (A, B). Show that $\mathcal{C}$ has a semi-additive structure, and explain briefly why this structure is unique.

Let $\mathbb{N}$ denote the multiplicative monoid of natural numbers (including 0), regarded as a category with one object. Show that $\mathbb{N}$ has infinitely many semi-additive structures. [*Hint: consider the monoid automorphisms of* $\mathbb{N}$.]

END OF PAPER