UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 5 June, 2017 1:30 pm to 4:30 pm

PAPER 115

DIFFERENTIAL GEOMETRY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet

Treasury Tag

Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Define what it means for $\pi : E \to M$ to be a *vector bundle* on a manifold M. Show that the tangent bundle

$$TM := \bigcup_p T_pM$$

with natural projection $\pi(v_p) = p$ for $v_p \in T_pM$ can be given the structure of a vector bundle on M.

(b) Suppose that $f : N \to M$ is a smooth map between manifolds N and M and $\pi : E \to M$ is a vector bundle on M. Show that

$$f^*E := \bigcup_{p \in N} \pi^{-1}(f(p))$$

with natural projection map $\pi(v_p) = p$ for $v_p \in E_{f(p)}$ can be given the structure of a vector bundle on N.

(c) If $\pi : E \to M$ and $\pi' : E' \to M$ are vector bundles over the same manifold, define what is meant by a *bundle morphism* between E and E' over M, and what it means for E and E' to be *isomorphic* as vector bundles over M. Prove that if $f : N \to M$ is a diffeomorphism then TN is isomorphic to f^*TM as vector bundles over N.

(d) Is it true that TM is always isomorphic to T^*M as vector bundles over M? Is it true that any two vector bundles on M of the same rank are isomorphic over M? Give a proof or counterexample.

2

(a) Suppose V is a real vector space and $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant l$. Denote by

$$C^i_j : \underbrace{V^* \otimes \cdots \otimes V^*}_{k} \otimes \underbrace{V \otimes \cdots \otimes V}_{l} \otimes \to \underbrace{V^* \otimes \cdots \otimes V^*}_{k-1} \otimes \underbrace{V \otimes \cdots \otimes V}_{l-1}$$

the contraction induced by the natural pairing between the i-th factor of V^* and the j-th factor of V. Show that, for suitable vector spaces V, the C^i_j induce a map

$$C^i_j : \mathcal{T}^k_l \to \mathcal{T}^{k-1}_{l-1}$$

where $\mathcal{T}^k_l$ denotes the space of smooth tensors of type (k, l) on a manifold M.

(b) Now suppose that $D : C^\infty(M) \to C^\infty(M)$ and $D : \mathrm{Vect}(M) \to \mathrm{Vect}(M)$ are $\mathbb{R}$-linear maps that satisfy

$$D(fY) = f \cdot DY + Df \cdot Y \text{ for } f \in C^\infty(M) \text{ and } Y \in \mathrm{Vect}(M).$$

Show that D has a unique extension, that we continue to denote by D, to a map $D : \mathcal{T}^k_l \to \mathcal{T}^k_l$ for all $k, l \geqslant 0$ such that

(i) D is $\mathbb{R}$-linear

(ii) $D(\alpha \otimes \beta) = D\alpha \otimes \beta + \alpha \otimes D\beta$ for all $\alpha \in \mathcal{T}^k_l$ and $\beta \in \mathcal{T}^{k'}_{l'}$

(iii) $DC^i_j = C^i_j D$ for any contraction $C^i_j : \mathcal{T}^k_l \to \mathcal{T}^{k-1}_{l-1}$.

[Hint: Start with defining D on 1-forms ω by demanding that if locally $D(\frac{\partial}{\partial x_i}) = \sum_j a_{ij} \frac{\partial}{\partial x_j}$ then $D(dx_i) := -\sum_j a_{ij} dx_j$ and also that $D(f\omega) = fD(\omega) + Df \cdot \omega$.]

(c) Now fix $X \in \mathrm{Vect}(M)$ and $A \in \mathcal{T}^1_1$. For $f \in C^\infty(X)$ and $Y \in \mathrm{Vect}(M)$ set

$$\begin{array}{ll} L_X(f) = X(f) & L_X(Y) = [X, Y] \\ D_A(f) = 0 & D_A(Y) = C^1_2(A \otimes Y). \end{array}$$

Show that both L_X and D_A extend uniquely to maps $\mathcal{T}^k_l \to \mathcal{T}^k_l$ satisfying properties (i), (ii) and (iii) from part (b).

(d) Finally prove that if $X \in \mathrm{Vect}(M)$ and $f \in C^\infty(M)$ then

$$D_{X \otimes df} = fL_X - L_{fX}.$$

3

(a) Let M be a compact connected manifold of dimension n. Define, in terms of differential forms, what it means for M to be *oriented*. Prove that if M is oriented then $H^n_{dR}(M)$ is non-zero.

(b) Now suppose that $f : M \to M$ is a diffeomorphism such that $f \circ f = id_M$. Prove that for all $p \geqslant 0$,

$$H^p_{dR}(M) = H^p_+(M) \oplus H^p_-(M)$$

where

$$H^p_+(M) := \{\alpha \in H^p_{dR}(M) : f^*\alpha = \alpha\}$$
$$H^p_-(M) := \{\alpha \in H^p_{dR}(M) : f^*\alpha = -\alpha\}.$$

(c) Now suppose that N is another manifold and $\pi : M \to N$ is a surjective smooth map, that M is covered by open sets U such that $\pi|_U : U \to \pi(U)$ is a diffeomorphism, and

$$\pi^{-1}(\pi(x)) = \{x, f(x)\} \text{ for all } x \in M.$$

Show that π^* induces an isomorphism

$$\pi^* : H^p_{dR}(N) \simeq H^p_+(M)$$

for all $p \geqslant 0$.

(d) Using this, or otherwise, compute $H^p_{dR}(\mathbb{RP}^n)$ for all $p \geqslant 0$ and $n \geqslant 1$.

[You may use without proof that the deRham cohomology space $H^p_{dR}(S^n)$ of the n-dimensional sphere is $\mathbb{R}$ for $p = 0, n$ and zero otherwise.]

4

(a) Define what is meant by a *linear connection* ∇ on a manifold M, the *curvature* of ∇, and what is meant by the *Levi-Civita* connection associated to a Riemannian metric g on M.

(b) Suppose that ∇^i is a linear connection on a manifold M_i for $i = 1, 2$. Show that there exists a linear connection ∇ on $M_1 \times M_2$ that satisfies

$$\nabla_{Y_1+Y_2}(X_1 + X_2) = \nabla^1_{Y_1}(X_1) + \nabla^2_{Y_2}(X_2)$$

for $X_i, Y_i \in \mathrm{Vect}(M_i)$, where $T_{(p,q)}(M_1 \times M_2)$ is identified with $T_pM_1 \oplus T_qM_2$.

(c) Now let g_i be a Riemannian metric on M_i for $i = 1, 2$. Show that g_i induce a Riemannian metric g on $M_1 \times M_2$. Show also that if ∇^i is the Levi-Civita connection for (M_i, g_i) for $i = 1, 2$ then ∇ is the Levi-Civita connection for $(M_1 \times M_2, g)$.

(d) Suppose that (M_i, g_i) are locally isometric to $\mathbb{R}^n$ with the Euclidean metric. Show that the curvature of the Levi-Civita connection on $(M_1 \times M_2, g)$ vanishes identically.

END OF PAPER