UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 2 June, 2017 1:30 pm to 4:30 pm

PAPER 114

ALGEBRAIC TOPOLOGY

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

For a continuous map $f : X \to X$, let $T_f = (X \times [0,1])/\sim$, where $(x,0) \sim (f(x),1)$. Show that there is a long exact sequence

$$\longrightarrow H_n(X;\mathbb{Z}) \xrightarrow{id-f_*} H_n(X;\mathbb{Z}) \longrightarrow H_n(T_f;\mathbb{Z}) \longrightarrow H_{n-1}(X;\mathbb{Z}) \xrightarrow{id-f_*} H_{n-1}(X;\mathbb{Z}) \longrightarrow .$$

An invertible 2×2 matrix A with integer entries gives a homeomorphism of $\mathbb{R}^2$ which preserves $\mathbb{Z}^2$, and hence defines a homeomorphism f_A of the torus $X = \mathbb{R}^2/\mathbb{Z}^2$. Show that the map $(f_A)_* : H_1(X;\mathbb{Z}) \to H_1(X;\mathbb{Z})$ is given by the matrix A, with respect to a basis which you should describe. Show that the map $(f_A)_* : H_2(X;\mathbb{Z}) \to H_2(X;\mathbb{Z})$ is given by multiplication by $\det(A)$. Hence compute the integral homology groups of T_{f_A} when

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

2

Without using cellular homology, carefully calculate the integral homology of S^n, making clear all results which you use. Let $f : S^n \to S^n$ be a continuous map. What is the *degree* of the map f? What is the *local degree* of the map f at a point $x_0 \in S^n$, and when is it defined? State and prove a result expressing the degree of f in terms of local degrees.

3

Define the canonical 1-dimensional real vector bundle $\pi : \gamma^{\mathbb{R}}_{1,n} \to \mathbb{RP}^n$, and prove that it is locally trivial. Describe the Gysin sequence associated to an $\mathbb{R}-$*oriented* vector bundle. Hence compute the cohomology ring $H^*(\mathbb{RP}^n;\mathbb{F}_2)$.

If $f : S^n \to S^m$ is a continuous map such that $f(-x) = -f(x)$, show that $n \leqslant m$.

If $g : S^n \times S^n \to S^n$ is a continuous map such that $g(-x,y) = -g(x,y) = g(x,-y)$, show that $n = 2^k - 1$ for some integer k.

4

Compute the effect of the inclusion map $i : \mathbb{RP}^2 \to \mathbb{CP}^2$ on $\mathbb{F}_2$-cohomology. By considering the pair $(\mathbb{CP}^{2k}, \mathbb{CP}^{2k} \setminus \mathbb{RP}^{2k})$, show that $\mathbb{CP}^{2k} \setminus \mathbb{RP}^{2k}$ has the same $\mathbb{F}_2$-cohomology ring as $\mathbb{CP}^{k-1} \times S^{2k}$. State carefully all results which you use.

5

Let $\mathbb{F}$ be a field. State the Poincaré duality theorem for a compact $\mathbb{F}$-oriented manifold M, and prove that it provides a non-singular bilinear form on $H^*(M;\mathbb{F})$.

Let W_g be a manifold obtained as the connect-sum of g copies of $S^{2n+1} \times S^{2n+1}$. Compute the integral cohomology ring of W_g, stating carefully all results which you use. If $f : W_g \to W_g$ is a smooth orientation preserving map such that $f \circ f = id_{W_g}$, and whose fixed points form a discrete set $\mathrm{Fix}(f)$, show that

$$\#\mathrm{Fix}(f) \equiv 2 - 2g \mod 4.$$

[You may assume that under the stated conditions on f one has $\det(I - D_x f) > 0$ for all $x \in \mathrm{Fix}(f)$. You may use that a finite dimensional real vector space which admits a non-singular skew-symmetric bilinear form is even dimensional.]

END OF PAPER