UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 7 June, 2017 1:30 pm to 4:30 pm

PAPER 113

ALGEBRAIC GEOMETRY

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

Throughout this paper, rings are commutative with element 1.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $\alpha\colon A \to B$ be a ring homomorphism and let $f\colon X = \text{Spec } B \to Y = \text{Spec } A$ be the induced morphism of schemes.

(i) Let $\phi\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ be the morphism on sheaves induced by α. Show that α is injective if and only if ϕ is injective.

(ii) Find an example in which α is not injective but f is a homeomorphism with respect to the Zariski topology.

(iii) Show that if X and Y are integral and if f is surjective, then the generic fibre of f is a non-empty integral scheme. [The generic fibre is the fibre of f over the generic point of Y.]

(iv) Find an example in which X is integral and there is $y \in Y$ such that the fibre of f over y is not reduced.

2

Let X be a scheme and let $\mathcal{I}$ be a quasi-coherent ideal sheaf on X.

(i) Explain carefully the construction of the closed subscheme of X associated to $\mathcal{I}$.

(ii) Show that there is no example in which X is Noetherian and reduced and $\mathcal{I}$ is locally free of rank two. [*Hint: First consider the case where X is irreducible; next reduce the problem to this case.*]

3

(i) Let X be an integral scheme. Show that each invertible sheaf $\mathcal{L}$ on X is isomorphic to $\mathcal{O}_X(D)$ for some Cartier divisor D on X.

(ii) Let X be a scheme and let $\mathcal{L}$ be an invertible sheaf on X. We say $\mathcal{L}$ is *generated by global sections* if for each point $x \in X$ there is $s \in \mathcal{L}(X)$ such that (X, s) generates $\mathcal{L}_x$ as an $\mathcal{O}_x$-module. Show that if $\mathcal{L}$ is generated by global sections, then $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}$ is also generated by global sections.

4

(i) Let X be a topological space and let $\mathcal{F}$ be a flasque sheaf on X. Show that $H^i(X, \mathcal{F}) = 0$ for every $i > 0$.

(ii) Let $\mathbb{P}^1_{\mathbb{C}} = \text{Proj } \mathbb{C}[t_0, t_1]$. Consider the open subscheme $D_+(t_0) \subset \mathbb{P}^1_{\mathbb{C}}$ and the inclusion morphism $f\colon D_+(t_0) \to \mathbb{P}^1_{\mathbb{C}}$. Show that if $\mathcal{G}$ is a quasi-coherent sheaf on $D_+(t_0)$, then $H^i(\mathbb{P}^1_{\mathbb{C}}, f_*\mathcal{G}) = 0$ for every $i > 0$.

5

Let $S = \mathbb{C}[t_0, \ldots, t_4]$ and let $\mathbb{P}^4_{\mathbb{C}} = \text{Proj } S$. Let $F_j \in S$ be a homogeneous polynomial of degree $d_j > 0$, for $j = 1, 2, 3$. Consider the ideals

$$I_1 = \langle F_1 \rangle, \; I_2 = \langle F_1, F_2 \rangle, \; I_3 = \langle F_1, F_2, F_3 \rangle$$

in S. Assume that for $j = 1, 2$ we have the following property:

if $G \in S$ is homogeneous and if $GF_{j+1} \in I_j$, then $G \in I_j$.

Let X_j be the closed subscheme of $\mathbb{P}^4_{\mathbb{C}}$ defined by the ideal sheaf $\tilde{I}_j$. Calculate $H^0(X_j, \mathcal{O}_{X_j})$ for $j = 1, 2, 3$.

[*Hint: Consider the exact sequences*

$$0 \to I_j/I_{j-1} \to S/I_{j-1} \to S/I_j \to 0$$

where we put $I_0 = 0$.]

END OF PAPER