PAPER 113

ALGEBRAIC GEOMETRY

Attempt no more than **FOUR** questions.

There are **FIVE** questions in total.

The questions carry equal weight.

*Throughout this paper, rings are commutative with element 1.*

**STATIONERY REQUIREMENTS**

- Cover sheet
- Treasury Tag
- Script paper

**SPECIAL REQUIREMENTS**

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.
Let $\alpha : A \to B$ be a ring homomorphism and let $f : X = \text{Spec } B \to Y = \text{Spec } A$ be the induced morphism of schemes.

(i) Let $\phi : O_Y \to f_*O_X$ be the morphism on sheaves induced by $\alpha$. Show that $\alpha$ is injective if and only if $\phi$ is injective.

(ii) Find an example in which $\alpha$ is not injective but $f$ is a homeomorphism with respect to the Zariski topology.

(iii) Show that if $X$ and $Y$ are integral and if $f$ is surjective, then the generic fibre of $f$ is a non-empty integral scheme. [The generic fibre is the fibre of $f$ over the generic point of $Y$.]

(iv) Find an example in which $X$ is integral and there is $y \in Y$ such that the fibre of $f$ over $y$ is not reduced.

Let $X$ be a scheme and let $\mathcal{I}$ be a quasi-coherent ideal sheaf on $X$.

(i) Explain carefully the construction of the closed subscheme of $X$ associated to $\mathcal{I}$.

(ii) Show that there is no example in which $X$ is Noetherian and reduced and $\mathcal{I}$ is locally free of rank two. [Hint: First consider the case where $X$ is irreducible; next reduce the problem to this case.]

(i) Let $X$ be an integral scheme. Show that each invertible sheaf $\mathcal{L}$ on $X$ is isomorphic to $O_X(D)$ for some Cartier divisor $D$ on $X$.

(ii) Let $X$ be a scheme and let $\mathcal{L}$ be an invertible sheaf on $X$. We say $\mathcal{L}$ is generated by global sections if for each point $x \in X$ there is $s \in \mathcal{L}(X)$ such that $(X, s)$ generates $\mathcal{L}_x$ as an $O_x$-module. Show that if $\mathcal{L}$ is generated by global sections, then $\mathcal{L} \otimes_{O_X} \mathcal{L}$ is also generated by global sections.

(i) Let $X$ be a topological space and let $\mathcal{F}$ be a flasque sheaf on $X$. Show that $H^i(X, \mathcal{F}) = 0$ for every $i > 0$.

(ii) Let $\mathbb{P}^1 = \text{Proj } \mathbb{C}[t_0, t_1]$. Consider the open subscheme $D_+(t_0) \subset \mathbb{P}^1$ and the inclusion morphism $f : D_+(t_0) \to \mathbb{P}^1$. Show that if $\mathcal{G}$ is a quasi-coherent sheaf on $D_+(t_0)$, then $H^i(\mathbb{P}^1, f_*\mathcal{G}) = 0$ for every $i > 0$. 

Part III, Paper 113
Let $S = \mathbb{C}[t_0, \ldots, t_4]$ and let $\mathbb{P}^4_C = \text{Proj } S$. Let $F_j \in S$ be a homogeneous polynomial of degree $d_j > 0$, for $j = 1, 2, 3$. Consider the ideals

$$I_1 = \langle F_1 \rangle, \ I_2 = \langle F_1, F_2 \rangle, \ I_3 = \langle F_1, F_2, F_3 \rangle$$

in $S$. Assume that for $j = 1, 2$ we have the following property:

if $G \in S$ is homogeneous and if $GF_{j+1} \in I_j$, then $G \in I_j$.

Let $X_j$ be the closed subscheme of $\mathbb{P}^4_C$ defined by the ideal sheaf $\tilde{I}_j$. Calculate $H^0(X_j, \mathcal{O}_{X_j})$ for $j = 1, 2, 3$.

[Hint: Consider the exact sequences

$$0 \to I_j/I_{j-1} \to S/I_{j-1} \to S/I_j \to 0$$

where we put $I_0 = 0$.]

END OF PAPER