UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 5 June, 2017 1:30 pm to 3:30 pm

PAPER 109

COMBINATORICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be a finite family of measurable subsets of $[0,1]$, and let $b_1, \ldots, b_n$ be positive reals. Show that there are disjoint measurable sets $B_1, \ldots, B_n$ with $B_i \subset A_i$ having measure $\lambda(B_i) = b_i$ if and only if

$$\lambda\Big(\bigcup_{i \in I} A_i\Big) \geqslant \sum_{i \in I} b_i$$

for all subsets I of $[n]$.

2

(i) State and prove the Erdős–Ko–Rado Theorem.

(ii) Two set systems, $\mathcal{A}$ and $\mathcal{B}$, are called *cross-intersecting* if $A \cap B \neq \emptyset$ whenever $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Show that there is a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that if $\mathcal{A} \subset \mathbb{N}^{(\leqslant r)}$ and $\mathcal{B} \subset \mathbb{N}^{(\leqslant s)}$ are cross-intersecting then there is a set X of at most $f(r,s)$ elements such that $A \cap B \cap X \neq \emptyset$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

3

(i) State and prove Harris's Lemma. If you deduce it from the Four Functions Theorem, you should prove that first.

(ii) Let $\mathcal{A}$ and $\mathcal{B}$ be non-empty *cross-Sperner* families of subsets of $[n]$, meaning that $\mathcal{A}$ and $\mathcal{B}$ are such that if $A \in \mathcal{A}$ and $B \in \mathcal{B}$ then neither $A \subset B$ nor $B \subset A$. [In particular, $\mathcal{A} \cap \mathcal{B} = \emptyset$.] Show that

$$|\mathcal{A}|^{1/2} + |\mathcal{B}|^{1/2} \leqslant 2^{n/2}.$$

(iii) Show that for $1 \leqslant k < 2k \leqslant n$ there are cross-Sperner families $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(n)$ such that $|\mathcal{A}| = 2^{-2k}2^n$ and $|\mathcal{B}| = \left(1 - 2^{-k}\right)^2 2^n$.

[Hint. For $K \subset [n]$, consider subsets of $[n]$ containing K, not containing K, meeting K and not meeting K.]

4

Let $0 \leqslant \ell_1 < \cdots < \ell_s < r < n$ be integers, $L = \{\ell_1, \ldots, \ell_s\}$, and let $\{A_1, \ldots, A_m\} \subset [n]^{(r)}$ be an L-intersecting family (i.e. $|A_i \cap A_j| \in L$ for all $1 \leqslant i < j \leqslant m$). Show that $m \leqslant \binom{n}{s}$.

END OF PAPER