UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 2 June, 2017 9:00 am to 12:00 pm

PAPER 108

TOPICS IN ERGODIC THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Define *unique ergodicity.*

Prove that irrational circle rotations are uniquely ergodic.

Let (X, T) be a uniquely ergodic system with unique invariant measure μ. Let $f \in C(X)$. Prove that for all $x \in X$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = \int f d\mu.$$

[You may use without proof the fact that averages of the pushforwards of a measure along orbit segments converge to an invariant measure.]

Denote by S the set of integers $n \in \mathbf{Z}_{\geqslant 0}$ such that the first digit of 2^n in its decimal expansion is 7. Prove that

$$\lim_{N\to\infty} \frac{|S \cap [0, N-1]|}{N} = \frac{\log 8 - \log 7}{\log 10}.$$

2

Define *mixing*, *convergence in density* and *weak mixing.*

Explain how the measure preserving system called the Chacón map is constructed.

[You do not need to prove that the map is well defined and measure preserving.]

Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system, let $f \in L^2(X, \mu)$ and suppose that

$$\lim_{n\to\infty} \langle U_T^n f, f\rangle = \Big|\int f d\mu\Big|^2.$$

Prove that

$$\lim_{n\to\infty} \langle U_T^n f, g\rangle = \int f d\mu \cdot \int \bar{g} d\mu$$

holds for all $g \in L^2(X, \mu)$. (Recall $U_T f = f \circ T$.)

[Hint: first prove the claim when $g = U_T^k f$ for some $k \in \mathbf{Z}_{\geqslant 0}$.]

Prove that a measure preserving system $(X, \mathcal{B}, \mu, T)$ is mixing if and only if

$$\lim_{n\to\infty} \mu(T^{-n}A \cap A) = \mu(A)^2$$

holds for all $A \in \mathcal{B}$.

3

Define *entropy* of a measure preserving system (include the definition of it *with respect to a partition*).

Prove that

$$h_\mu(T,\xi) = \lim_{n\to\infty} H_\mu(\xi|\xi_1^n).$$

State the *Kolmogorov-Sinai theorem.*

Calculate the entropy of Bernoulli shifts.

Let $(X,\mathcal{B},\mu,T)$ be an invertible measure preserving system and let $\xi \subset \mathcal{B}$ be a finite partition. Suppose that for every $\varepsilon > 0$ and for every set $A \in \mathcal{B}$, there is a number $n \in \mathbf{Z}_{\geqslant 0}$ and there is $B \in \mathcal{B}(\xi_0^{n-1})$ such that $\mu(A \triangle B) < \varepsilon$. (A partition with this property is called a one-sided generator.) Prove that $h_\mu(T) = 0$.

[You may use without proof any result contained in the lectures or in the example sheets.]

4

Let $(X,\mathcal{B},\mu,T)$ be a measure preserving system. Suppose that $h_\mu(T,\xi) > 0$ for all finite partitions $\xi \subset \mathcal{B}$ that satisfy $H_\mu(\xi) > 0$. Prove that the tail σ-algebra $\mathcal{T}(\xi)$ is trivial for all finite partitions $\xi \subset \mathcal{B}$.

END OF PAPER