UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 12 June, 2017 9:00 am to 12:00 pm

PAPER 107

ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let Ω be a bounded domain in $\mathbb{R}^n$.

(a) Let $L = a^{ij}D_{ij} + b^i D_i + c$ be an elliptic operator on Ω. Giving the additional hypotheses needed, state and prove the weak maximum principle for a function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying $Lu \geqslant 0$ in Ω.

(b) If $w \in C^\infty(\Omega) \cap C^1(\overline{\Omega})$ is harmonic in Ω, show that

$$\sup_{\Omega} |Dw| = \sup_{\partial\Omega} |Dw|.$$

(c) If $v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies $\Delta v = v^3 - v$ in Ω, and if $\sup_{\partial\Omega} |v| < 1$, show that $\sup_{\Omega} |v| \leqslant 1$.

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(a) Let $L = a^{ij}D_{ij} + b^i D_i$ be an elliptic operator on a domain $\Omega \subset \mathbb{R}^n$, and let $y \in \partial\Omega$. Giving the additional hypotheses needed, state and prove the Hopf boundary point lemma for a function $u \in C^2(\Omega) \cap C^0(\Omega \cup \{y\})$ satisfying $Lu \geqslant 0$ in Ω and $u(y) > u(x)$ for all $x \in \Omega$.

[You may use without proof the weak maximum principle.]

(b) Let $\Omega \subset \mathbb{R}^n$ be a domain and let $v \in C^1(\Omega)$ be non-positive. Let $Z = \{x \in \Omega : v(x) = 0\}$. Use the Hopf boundary point lemma to show that if $v \in C^2(\Omega \setminus Z)$ and satisfies $\Delta v + v = 0$ in $\Omega \setminus Z$, then either $v(x) = 0$ for every $x \in \Omega$ or $v(x) < 0$ for every $x \in \Omega$.

Give an example to show that this conclusion does not hold if in place of the hypothesis $v \in C^1(\Omega)$ it is assumed that v is locally Lipschitz in Ω, and all other hypotheses are kept unchanged.

3 Let $\alpha \in (0,1)$ and let Ω be a bounded $C^{2,\alpha}$ domain in $\mathbb{R}^n$. Let $c \in C^{0,\alpha}(\overline{\Omega})$.

(a) State without proof the global Schauder estimate satisfied by a function $u \in C^{2,\alpha}(\overline{\Omega})$ solving the Dirichlet problem

$$\begin{aligned} \Delta u + cu &= f \text{ in } \Omega, \\ u &= \psi \text{ on } \partial\Omega \end{aligned} \qquad (\star)$$

where $f \in C^{0,\alpha}(\overline{\Omega})$ and $\psi \in C^{2,\alpha}(\overline{\Omega})$.

If $c \leqslant 0$ in Ω, use the weak maximum principle to show that

$$\sup_{\Omega} |u| \leqslant \sup_{\partial\Omega} |\psi| + C \sup_{\Omega} |f|$$

for some constant C depending only on Ω.

[*Hint: Choose* $d > 0$ *such that* $\Omega \subset \{-d < x_1 < d\}$ *and consider the function* $v = \sup_{\partial\Omega} |\psi| + (e^{2d} - e^{x_1+d}) \sup_{\Omega} |f|$.]

(b) State and prove the Fredholm alternative concerning solvability for $u \in C^{2,\alpha}(\overline{\Omega})$ of the Dirichlet problem $(\star)$ for given functions $f \in C^{0,\alpha}(\overline{\Omega})$ and $\psi \in C^{2,\alpha}(\overline{\Omega})$.

[*You may use without proof any theorem from abstract functional analysis and any standard existence theorem for solutions to elliptic equations.*]

(c) Show that there exists a constant $\epsilon = \epsilon(\Omega) > 0$ such that if $c(x) \leqslant \epsilon$ for all $x \in \Omega$, then for any given $f \in C^{0,\alpha}(\overline{\Omega})$ and $\psi \in C^{2,\alpha}(\overline{\Omega})$, the Dirichlet problem $(\star)$ has a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$.

4 Let $B_1(0)$ be the open unit ball in $\mathbb{R}^n$ and let $\alpha \in (0,1)$. Show that for each $\delta \in (0,1)$, there is a constant $C = C(n,\alpha,\delta) \in (0,\infty)$ such that if $u \in C^{2,\alpha}(\overline{B_1(0)})$ is a solution to $\Delta\, u = f$ in $B_1(0)$ for some $f \in C^{0,\alpha}(\overline{B_1(0)})$, then

$$[D^2u]_{\alpha;\, B_{1/2}(0)} \leqslant \delta[D^2u]_{\alpha;\, B_1(0)} + C\left(|u|_{2;\, B_1(0)} + |f|_{0,\alpha;\, B_1(0)}\right).$$

[*You may use without proof Liouville's Theorem: there does not exist a non-constant harmonic function w on $\mathbb{R}^n$ such that $[w]_{\alpha;\,\mathbb{R}^n} < \infty$.*]

Explain briefly how to deduce from the above result that there is a constant $C = C(n,\alpha)$ such that, for u and f as above,

$$[D^2u]_{\alpha;B_{1/2}(0)} \leqslant C\left(|u|_{2;\, B_1(0)} + |f|_{0,\alpha;\, B_1(0)}\right). \qquad (\star)$$

[*You are not required to give proofs of any additional results needed.*]

Give an example to show that the estimate $(\star)$ cannot be improved to

$$[D^2u]_{\alpha;B_{1/2}(0)} \leqslant C\left(|u|_{2;\, B_1(0)} + |f|_{0;\, B_1(0)}\right)$$

for some constant $C = C(n,\alpha)$.

5 Let $B_1(0)$ be the open unit ball in $\mathbb{R}^n$, and let $a^{ij} \in L^\infty(B_1(0))$ be such that $\|a^{ij}\|_{L^\infty(B_1(0))} \leqslant \Lambda$ and $a^{ij}(x)\zeta^i\zeta^j \geqslant \lambda|\zeta|^2$ for some constants $\Lambda \geqslant \lambda > 0$, all $\zeta \in \mathbb{R}^n$ and a.e. $x \in B_1(0)$. Let $Lu = D_i\left(a^{ij}D_j u\right)$.

(a) Let $u \in W^{1,2}(B_1(0))$ be a non-negative weak supersolution of $Lu = 0$ in $B_1(0)$. State without proof the *weak Harnack inequality* giving a lower bound for $\inf_{B_{1/2}(0)} u$.

(b) Let $u \in W^{1,2}(B_1(0))$ be a weak solution of $Lu = 0$ in $B_1(0)$. Assuming that $u \in L^\infty(B_1(0))$, use the estimate in (a) to show that $u \in C^{0,\mu}\left(\overline{B_{1/4}(0)}\right)$, and that

$$\sup_{x,y\in B_{1/4}(0),\, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\mu} \leqslant C\|u\|_{L^\infty(B_1(0))}$$

for some constants $\mu \in (0,1)$ and $C > 0$ depending only on n, λ and Λ.

(c) Let $u_k \in W^{1,2}(B_1(0)) \cap C^0(\overline{B_1(0)})$ be a non-zero weak solution of $Lu = 0$ in $B_1(0)$ for each $k = 1, 2, 3, \ldots$. Show that there is a function $v \in W^{1,2}_{\text{loc}}(B_1(0)) \cap C^0(B_{1/4}(0))$ satisfying $Lv = 0$ weakly in $B_\theta(0)$ for every $\theta \in (0,1)$ and a subsequence $u_{k'}$ such that

$$\frac{u_{k'}}{\left(\sup_{B_1(0)} |u_{k'}|\right)} \to v$$

uniformly on $B_{1/4}(0)$ as $k' \to \infty$.

[*Hint: Show first that* $\int |Du_k|^2\varphi^2 \leqslant C\int |u_k|^2|D\varphi|^2$ *for some constant* C *independent of* k *and any* $\varphi \in C^1_c(B_1(0))$.]

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(a) Let $B_1(0)$ be the open unit ball in $\mathbb{R}^n$ and let $Lu = D_i\left(a^{ij}D_ju\right)$ where $a^{ij} \in L^\infty(B_1(0))$, $\|a^{ij}\|_{L^\infty(B_1(0))} \leqslant \Lambda$ and $a^{ij}(x)\zeta^i\zeta^j \geqslant \lambda|\zeta|^2$ for some constants $\Lambda \geqslant \lambda > 0$, all $\zeta \in \mathbb{R}^n$ and a.e. $x \in B_1(0)$. Let $u \in W^{1,2}(B_1(0)) \cap L^\infty(B_1(0))$ and suppose that u is a non-negative weak subsolution of $Lu = 0$ in $B_1(0)$.

Show that

$$\int_{B_1(0)} |Du|^2 u^{\alpha-2}\eta^2 \leqslant \frac{C}{(\alpha-1)^2}\int_{B_1(0)} u^\alpha |D\eta|^2$$

for each $\alpha > 1$ and $\eta \in C^1_c(B_1(0))$, where $C = C(\lambda, \Lambda)$.

For $n \geqslant 3$, the Sobolev inequality says that

$$\left(\int_{B_1(0)} f^{2\sigma}\right)^{\frac{1}{\sigma}} \leqslant C\int_{B_1(0)} |Df|^2$$

for every $f \in W^{1,2}_0(B_1(0))$, where $\sigma = \frac{n}{n-2}$ and $C = C(n)$. Use the Sobolev inequality and the above estimate to show that for $n \geqslant 3$, $0 < r < R \leqslant 1$ and any $p > 1$,

$$\left(\int_{B_r(0)} u^{\sigma\alpha}\right)^{\frac{1}{\sigma\alpha}} \leqslant \frac{C^{\frac{1}{\alpha}}}{(R-r)^{\frac{2}{\alpha}}}\left(\int_{B_R(0)} u^\alpha\right)^{\frac{1}{\alpha}}$$

for any $\alpha \geqslant p$, where $C = C(n, p, \lambda, \Lambda)$.

Deduce that for $n \geqslant 3$ and any $p > 1$,

$$\|u\|_{L^\infty(B_{1/2}(0))} \leqslant C\|u\|_{L^p(B_1(0))}$$

for some constant $C = C(n, p, \lambda, \Lambda)$.

(b) Let $n \geqslant 3$ and now suppose that $a^{ij} \in L^\infty(\mathbb{R}^n)$, $\|a^{ij}\|_{L^\infty(\mathbb{R}^n)} \leqslant \Lambda$ and that $a^{ij}(x)\zeta^i\zeta^j \geqslant \lambda|\zeta|^2$ for some constants $\Lambda \geqslant \lambda > 0$, all $\zeta \in \mathbb{R}^n$ and a.e. $x \in \mathbb{R}^n$. Let $Lu = D_i\left(a^{ij}D_ju\right)$. If $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$, $u \geqslant 0$ and u is a weak subsolution of $Lu = 0$ in $B_R(0)$ for every $R > 0$, and if $\int_{\mathbb{R}^n} u^p < \infty$ for some $p > 1$, show that $u \equiv 0$ in $\mathbb{R}^n$.

END OF PAPER