UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 1 June, 2017 1:30 pm to 4:30 pm

PAPER 106

FUNCTIONAL ANALYSIS

*Attempt no more than **THREE** questions,*

*with at most **TWO** questions from each section.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

State the Hahn–Banach theorem on the extension of bounded linear functionals. Use it to prove that for a normed space X and for any non-zero $x \in X$ there exists $f \in X^*$ with $\|f\| = 1$ and $f(x) = \|x\|$. Deduce that X embeds isometrically into X^{**}.

Let X be a normed space. Let Γ be an arbitrary set, and let $\ell_\infty(\Gamma)$ be the Banach space of bounded real-valued functions on Γ with the supremum norm. Prove that a map $T\colon X \to \ell_\infty(\Gamma)$ is a bounded linear operator if and only if there is a bounded set $\{f_\gamma \mid \gamma \in \Gamma\} \subset X^*$ such that $Tx = (f_\gamma x)_{\gamma\in\Gamma}$, and in this case $\|T\| = \sup_{\gamma\in\Gamma}\|f_\gamma\|$.

Prove that for any normed space X there is a set Γ and an isometric embedding $T\colon X \to \ell_\infty(\Gamma)$. Show that if X is separable or if X is the dual of a separable space, then one can take $\Gamma = \mathbb{N}$.

Given $\lambda \geqslant 1$, we say a normed space X is λ-*injective* if whenever Y is a subspace of a normed space Z and $T \in \mathcal{B}(Y, X)$, there exists $\tilde{T} \in \mathcal{B}(Z, X)$ such that $\tilde{T}\restriction_Y = T$ and $\|\tilde{T}\| \leqslant \lambda\|T\|$. Prove that $\ell_\infty(\Gamma)$ is 1-injective for any set Γ.

Given $\lambda \geqslant 1$, prove that X is λ-injective if and only if whenever $T\colon X \to Z$ is an isometric embedding, there is a bounded linear map $P\colon Z \to X$ such that $\|P\| \leqslant \lambda$ and $P \circ T$ is the identity on X. [*Hint: for the "only if" part consider* $Y = T(X)$ *and a suitable map* $Y \to X$*; for the "if" part first embed* X *isometrically into some* $\ell_\infty(\Gamma)$.]

2

(a) Define the *weak topology* of a normed space. State and prove Mazur's theorem. [Any version of the Hahn–Banach theorem can be used without proof.] Define what it means for a subset of a normed space to be *weakly bounded*. Prove that a weakly bounded set is bounded in norm. Deduce that every weakly compact set is bounded in norm.

Let X be a Banach space. Prove that X is reflexive if and only if the closed unit ball B_X is weakly compact. [You may use without proof any results about the w*-topology of a dual space provided you clearly state them.]

Let X be a Banach space. Assume that the closed unit ball B_{X^*} of X^* contains a countable subset $\{f_n \mid n \in \mathbb{N}\}$ that separates the points of X. Prove that the weak topology on any weakly compact subset of X is metrizable. [You don't need to check all properties of a metric.]

(b) Using the results in part (a), prove the following.

(i) Let (f_n) be a sequence in $C[0,1]$ that converges weakly to zero. Prove that f_n converges to zero in $L_1[0,1]$ (in the L_1-norm).

(ii) Let X be a reflexive Banach space and C a non-empty, closed, convex subset of X. Show that there exists $x \in C$ such that $\|x\| = \inf_{y \in C} \|y\|$.

(iii) Prove that a weakly compact subset of ℓ_∞ is norm-separable.

3

Define the w*-topology on the dual space of a normed space. Show that if X is an infinite-dimensional Banach space, then the w*-topology on X^* is not metrizable. [*Hint: Prove that if there are w*-neighbourhoods U_n, $n \in \mathbb{N}$, of 0 in X^* such that every w*-neighbourhood V of 0 contains U_n for some n, then X has a countable basis. You may use elementary results from linear algebra without proof.*]

State and prove the theorems of Banach–Alaoglu and Goldstein. Prove that for every normed space X there is a compact Hausdorff space K such that X isometrically embeds into $C(K)$. [Results from general topology and any version of the Hahn–Banach theorem can be used without proof.]

Let X be a Banach space whose dual X^* is separable. Prove that every bounded sequence (x_n) in X has a subsequence (y_n) such that $\varphi(f) = \lim_{n\to\infty} f(y_n)$ exists for all $f \in X^*$, and show that $\varphi \in X^{**}$. Assuming in addition that X is not reflexive, show that there is a bounded sequence (x_n) in X no subsequence of which converges weakly in X. [No result can be used without proof.]

SECTION B

4

(a) Define the terms *character* and *character space* for a Banach algebra. Identify the character space of $R(K)$, where K is a nonempty compact subset of $\mathbb{C}$ and $R(K)$ is the closure in $C(K)$ of the rational functions without poles in K.

(b) Let A be a commutative unital Banach algebra and let $x \in A$. Prove that

$$\sigma_A(x) = \{\varphi(x) \mid \varphi \in \Phi_A\} .$$

[*You may assume without proof the Gelfand–Mazur theorem on complex unital normed division algebras and that the group of invertible elements of A is open.*]

Let U be an open subset of $\mathbb{C}$ with $\sigma_A(x) \subset U$. State the Holomorphic Functional Calculus for A, x and U. Given a holomorphic function $f \colon U \to \mathbb{C}$, denote by $f(x)$ the element of A produced by the Holomorphic Functional Calculus for f. Prove that $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Fix a holomorphic function $f \colon U \to \mathbb{C}$. State the formula that defines the element $f(x)$ of A. Deduce that

$$\sigma_A(f(x)) = \{f(\lambda) \mid \lambda \in \sigma_A(x)\} .$$

[*Standard properties of vector-valued integrals can be used without proof.*]

Let V be an open subset of $\mathbb{C}$ that contains $f(U)$. Show that $\sigma_A(f(x)) \subset V$ and that for every holomorphic function $g \colon V \to \mathbb{C}$ we have $g(f(x)) = (g \circ f)(x)$. Deduce, or otherwise prove, that if $\|x\| < 1$, then $1 - x$ is an *exponential*: $1 - x = \exp(y)$ for some $y \in A$.

(c) State and prove Runge's theorem on the approximation of holomorphic functions by rational functions with prescribed sets of poles. [You may of course assume all results in parts (a) and (b) to answer this part as well as any result from elementary spectral theory of Banach algebras.]

5

State the Riesz Representation Theorem identifying the dual space of the complex Banach space $C(K)$ for a compact Hausdorff space K.

[*For the rest of this question you may assume that if H is a non-zero complex Hilbert space, K is a compact Hausdorff space, and P is a resolution of the identity of H over K, then there is a unique isometric unital $*$-homomorphism $f \mapsto \int_K f\,dP \colon L_\infty(P) \to \mathcal{B}(H)$ such that $\left\langle \left(\int_K f\,dP\right) x, y\right\rangle = \int_K f\,P_{x,y}$ for every $f \in L_\infty(P)$, $x, y \in H$. Further, $\left\|\left(\int_K f\,dP\right) x\right\|^2 = \int_K |f|^2\,dP_{x,x}$ for every $f \in L_\infty(P)$, $x \in H$.*]

Prove the Spectral Theorem for commutative unital C*-algebras: Let H be a non-zero complex Hilbert space, let A be a commutative, unital C*-subalgebra of $\mathcal{B}(H)$, and let $K = \Phi_A$. Then there is a unique resolution P of the identity of H over K such that

$$T = \int_K \hat{T}\,dP \qquad \text{for every } T \in A\ ,$$

where $\hat{T}$ is the Gelfand transform of T. [You may assume the Gelfand–Naimark theorem for commutative unital C*-algebras without proof.] Prove also that $P(U) \neq 0$ for every non-empty open subset U of K.

State the Spectral Theorem for normal operators. Give a *brief* sketch of the proof.

Let H be a non-zero complex Hilbert space and let $T \in \mathcal{B}(H)$ be a normal operator. Assume that the spectrum $K = \sigma(T)$ has at least two points. Prove that T has a nontrivial invariant subspace: There is a closed non-zero proper subspace Y of H such that $Ty \in Y$ for every $y \in Y$. [*Hint: construct a suitable orthogonal projection Q with $TQ = QT$.*]

END OF PAPER