UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 5 June, 2017 9:00 am to 12:00 pm

PAPER 105

ANALYSIS OF PARTIAL DIFFERENTIAL EQUATIONS

Attempt all ***THREE*** *questions.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

1. State the Cauchy–Kovalevskaya Theorem (CKT) for general order systems of partial differential equations. Explain briefly why the proof reduces to the case of flat Cauchy hypersurfaces and first-order systems.

   Consider now the following three examples:

   (a) the Laplace equation $\Delta_x u = 0$ on $\mathbb{R}^n$ with Cauchy hypersurface $\{x_1 = 0\}$

   (b) the wave equation $(-\partial_t^2 + \Delta_x)u = 0$ on $\mathbb{R}^{n+1}$ with Cauchy hypersurface $\{t = 0\}$

   (c) the heat equation $(-\partial_t + \Delta_x)u = 0$ on $\mathbb{R}^{n+1}$ with Cauchy hypersurface $\{t = 0\}$.

   To which of these examples does the CKT apply? Briefly contrast this to the issue of well-posedness.

2. Let $B(0,1) \subset \mathbb{R}^2$ denote the open unit ball around the origin, let $a_\alpha(x,y)$ be analytic functions on $B(0,1)$ and consider the m-th order *linear* partial differential operator

$$P := \sum_{|\alpha| \leqslant m} a_\alpha(x,y)\partial^\alpha.$$

   Assume that the hypersurface $\Sigma \doteq \{x = 0\}$ is non-characteristic for P in $B(0,1)$. The aim of this problem will be to prove a version of *Holmgren's uniqueness theorem* in this context, i.e. the statement that C^m solutions u of the Cauchy problem for $Pu = 0$ with trivial data on Σ must vanish in a neighbourhood of the origin.

   (a) Write explicitly what it means for Σ above to be non-characteristic for P.

   (b) Show that there is a unique linear partial differential operator P^* such that for any open $\Omega \subset B(0,1)$ and any $u \in C^m(\Omega)$ and $v \in C_c^m(\Omega)$ (i.e. v with compact support) one has

$$\int_\Omega Pu \cdot v \,\mathrm{d}x = \int_\Omega u \cdot P^* v \,\mathrm{d}x. \tag{1}$$

   (c) Prove that (1) still holds when $u, v \in C^m(\overline{\Omega})$ and for each $x \in \partial\Omega$ and any β multi-index with $|\beta| \leqslant m-1$ either $\partial^\beta u(x) = 0$ or $\partial^\beta v(x) = 0$.

   (d) Prove that Σ is non-characteristic for P^* in $B(0,1)$. For $\epsilon \in (0,1)$, define $\tilde{\Sigma}_\epsilon = \{x = \epsilon - y^2\} \cap \{x > 0\}$. Show that for all sufficiently small ϵ, the curve $\tilde{\Sigma}_\epsilon$ is non-characteristic for P^*.

   (e) For $\epsilon > 0$, consider the transformation $\tilde{x}_\epsilon = x + y^2 - \epsilon$, $\tilde{y}_\epsilon = y$. Show that if $f(x,y)$ is an analytic function on $B(0,1)$, then for sufficiently small ϵ_0, the collection of functions $\tilde{f}_\epsilon(\tilde{x},\tilde{y}) := f_\alpha(x + y^2 - \epsilon, y)$ where $\epsilon \in (0,\epsilon_0)$ are well defined in the ball $\tilde{B}(0,1/2)$ of radius $1/2$ as functions of $\tilde{x}$ and $\tilde{y}$, and satisfy uniform analyticity bounds in those variables, i.e. show that there exist constants $\gamma, \zeta > 0$ such that for all $\epsilon \in (0,\epsilon_0)$, we have

$$\|\partial^\beta \tilde{f}_\epsilon\|_{L^\infty(\tilde{B}(0,1/2))} \leqslant \gamma \beta! \zeta^{|\beta|}.$$

(f) Let $g(x, y)$ be an analytic function on $B(0, 1)$. Consider for $\epsilon \in (0, 1)$ the curve $\tilde{\Sigma}_\epsilon$ as above, and the Cauchy problem,

$$P^* v = g \quad \text{and} \quad \partial^\beta v_{|\tilde{\Sigma}_\epsilon} = 0 \text{ for all } |\beta| \leqslant m - 1. \tag{2}$$

Using the CKT, (d) and (e), prove that there exists an ϵ, depending on g, such that there is an analytic solution v of (2) defined in the entire region $\omega_\epsilon := \{x > 0 \text{ and } x + y^2 < \epsilon\} \subset B(0, 1)$. [You may use the fact that, rewriting (2) in coordinates $(\tilde{x}, \tilde{y})$ as an equation $\tilde{P}^* \tilde{v} = \tilde{g}$, the proof of the CKT by the method of majorants gives a radius of analyticity on the solution $\tilde{v}$ which depends only on uniform analyticity bounds for the coefficients of $\tilde{P}^*$ and $\tilde{g}$.]

(g) Let $\mathcal{P}$ denote the set of polynomials in the variables (x, y). Show that restricting to $\mathcal{P}$, one can choose ϵ in (f) independent of $g \in \mathcal{P}$, i.e. prove that there exists an $\epsilon > 0$ such that *for all* $g \in \mathcal{P}$, there is an analytic solution (2) defined in the entire region ω_ϵ. [*Hint: Note that the transformed $\tilde{g}$ is again polynomial and use this fact together with the linearity of the equation to remove the dependence on analyticity bounds of $\tilde{g}$ in (f).*]

(h) Consider now a C^m solution u on $B(0, 1)$ to

$$Pu = 0 \quad \text{and} \quad \partial^\beta u_{|\Sigma} = 0 \text{ for all } |\beta| \leqslant m - 1. \tag{3}$$

Using (1) and the Weierstrass approximation theorem, prove that $u = 0$ in ω_ϵ for ϵ as in (g), and conclude that $u = 0$ on a neighbourhood of the origin. [The Weierstrass approximation theorem states that polynomials are dense in the space $C(X)$ where $X \subset \mathbb{R}^n$ is compact.]

2

Let $X = \{\frac{1}{2} \leqslant |x| \leqslant 2\} \subset \mathbb{R}^3$ denote an annulus, and consider the Dirichlet problem

$$\frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial u}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(g(|x|)\frac{\partial u}{\partial x_3}\right) = f \tag{1}$$

$$u|_{\partial X} = 0 \tag{2}$$

where $g : \mathbb{R} \to \mathbb{R}$ denotes a smooth function.

(a) Write down what it means for $u \in H^1(X)$ to satisfy weakly (1)–(2), for $f \in L^2(X)$.

(b) Show that if $g : \mathbb{R} \to \mathbb{R}$ is assumed in addition to be positive, then the equation (1) is uniformly elliptic on X.

(c) Let u be a weak solution of (1)–(2) as in (a) with g as in (b). Prove that there exists a constant $C > 0$ independent of u and f such that

$$\|u\|_{H^1(X)} \leqslant C\|f\|_{L^2(X)}. \tag{3}$$

Infer the uniqueness of weak solutions.

(d) Suppose now that $g : \mathbb{R} \to \mathbb{R}$ is smooth but <u>not</u> necessarily positive. Show that (3) does not necessarily hold for a uniform constant C independent of u and f.

(e) Define the usual spherical coordinates (r, θ, ϕ) on $\mathbb{R}^3$ by the relations $x_3 = r\cos\theta$, $x_1 = r\cos\phi\sin\theta$, $x_2 = r\sin\phi\sin\theta$. Show that the standard Laplacian is given by

$$\Delta_x u = \frac{1}{r^2}\partial_r(r^2\partial_r u) + \frac{1}{r^2\sin^2\theta}(\partial_\phi^2 u) + \frac{1}{r^2\sin\theta}(\partial_\theta(\sin\theta\partial_\theta u)) \tag{4}$$

and that the volume form is given by

$$dx_1\, dx_2\, dx_3 = r^2\sin\theta\, dr\, d\theta\, d\phi$$

in these coordinates.

(f) Use the representation (4) to give a direct proof that if $u \in C^\infty(X)$ is a classical solution of (1)–(2) <u>with $g = 1$</u> (i.e. u satisfies $\Delta_x u = f$) and $f \in C^\infty(X)$, then

$$\|u\|_{H^2(X)} \leqslant C\|f\|_{L^2(X)}. \tag{5}$$

[*Hint: Multiply by $\partial_\phi^2 u$ and by $\partial_\theta(\sin\theta\partial_\theta u)$ and integrate by parts with respect to the volume form. What happens with the boundary terms? How does one eventually estimate $(\partial_r^2 u)^2$?*]

(g) Let u be as in (f). Show then that $\partial_\phi u$ is again a classical $C^\infty(X)$ solution of (1)–(2) with $g = 1$ and with right hand side $\partial_\phi f$. Show the same statement where the coordinates (r, θ, ϕ) are redefined permuting the roles of x_1, x_2 and x_3 in (e). Now suppose that f is radial, i.e. it is a function of $f(|x|)$. Show that u is also radial, i.e. $u = u(|x|)$.

(h) Let u be again as in (f). Show that

$$\|u\|_{L^\infty(X)} \leqslant C\|f\|_{L^2(X)}$$

for a constant C independent of u and f. Now suppose f is radial. Show that

$$\|\nabla u\|_{L^\infty(X)} \leqslant C\|f\|_{L^2(X)} \tag{6}$$

for a constant C independent of u and f. Show in contrast that the inequality (6) does *not* hold for a uniform constant C if the assumption of radiality on f is dropped.

3

1. Consider the following Cauchy problem for an unknown real function $u : \mathbb{R}^d \to \mathbb{R}$:

$$\begin{cases} \partial_t u(t,x) + \mathbf{F}(t,x) \cdot \nabla_x u(t,x) = 0, & t \in \mathbb{R},\ x \in \mathbb{R}^d \\ u(t=0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1}$$

where $\mathbf{F} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is C^1 and satisfies $\sup_{\mathbb{R}\times\mathbb{R}^d} \frac{\|F(t,x)\|}{1+\|x\|} < +\infty$.

(a) Define the *characteristics* of this equation and explain why they are globally defined.

(b) State carefully the existence and uniqueness of C^1 solutions when u_0 is C^1. State carefully the existence and uniqueness of L^∞ weak solutions when u_0 is L^∞ (the notion of weak solutions must be defined). What is the weak-strong uniqueness principle?

(c) What can happen when $\mathbf{F}$ depends on the unknown u?

2. Consider now the following Cauchy problem for an unknown real function $u : \mathbb{R} \to \mathbb{R}$:

$$\begin{cases} \partial_t u + \partial_x [f(u)] = 0, & t \geqslant 0,\ x \in \mathbb{R} \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{2}$$

where $f : \mathbb{R} \to \mathbb{R}$ is C^1 and its derivative f' is L^∞ on $\mathbb{R}$.

(a) State the definition of weak solution for the Cauchy problem (2). State the Rankine–Hugoniot condition characterizing piecewise constant weak solutions. Give an example of non-uniqueness of weak solutions.

(b) We recall that an *entropic solution* to (2) is $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ such that for all $\varphi \in C^1_c(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R}_+)$ all $\eta : \mathbb{R} \to \mathbb{R}$ convex and piecewise C^1 and $\psi : \mathbb{R} \to \mathbb{R}$ an antiderivative of $f'\eta'$,

$$\int_0^T \int_{\mathbb{R}} \big(\eta(u)\partial_t \varphi + \psi(u)\partial_x \varphi\big)\, \mathrm{d}t\, \mathrm{d}x + \int_{\mathbb{R}} \varphi(0,x)\eta(u_0(x))\, \mathrm{d}x \geqslant 0. \tag{3}$$

Prove that a classical C^1 solution is an entropic solution, and that an entropic solution is a weak solution.

(c) Consider $\eta_k(u) := |u-k|$ for $k \in \mathbb{R}$. Calculate an associate flux $\psi(u)$ that satisfies the condition (3) above.

(d) Consider u and v two entropic solutions with initial data u_0 and v_0. Use η_k with $k = v(s,y)$ for u and η_k with $k = u(t,x)$ for v to establish

$$\begin{aligned} 0 \leqslant &\int_0^T \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} |u(t,x) - v(s,y)| (\partial_t \Phi + \partial_s \Phi)\, \mathrm{d}t\, \mathrm{d}s\, \mathrm{d}x\, \mathrm{d}y \\ &+ \int_0^T \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{sgn}(u(t,x) - v(s,y))[f(u(t,x)) - f(v(s,y))](\partial_x \Phi + \partial_y \Phi)\, \mathrm{d}t\, \mathrm{d}s\, \mathrm{d}x\, \mathrm{d}y \\ &+ \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} |u_0(x) - v(s,y)| \Phi(0,x,s,y)\, \mathrm{d}s\, \mathrm{d}x\, \mathrm{d}y \\ &+ \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} |u(t,x) - v_0(y)| \Phi(t,x,0,y)\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}y. \end{aligned}$$

for any test function $\Phi(t,x,s,y) \in C^1_c(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}; \mathbb{R}_+)$.

(e) Consider $\varphi \in C_c^1(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R}_+)$ and choose

$$\Phi(t,x,s,y) := \varphi(t,x)\chi_\varepsilon(t-s,x-y), \ \chi_\varepsilon(\tau,z) := \varepsilon^{-2}\chi\left(\frac{\tau}{\varepsilon},\frac{z}{\varepsilon}\right), \ \chi(\tau,z) := \zeta(\tau)\theta(z)$$

with $\zeta \geqslant 0$ smooth with $\int_{-\infty}^{\infty} \zeta(\tau)d\tau = 1$ and support in $[-2,-1]$ and $\theta \geqslant 0$ smooth with $\int_{-\infty}^{\infty} \theta(z) = 1$ and compact support. By studying the limit $\varepsilon \to 0$ in the previous integral inequality deduce

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}} |u(t,x)-v(t,x)|\partial_t\varphi \,\mathrm{d}t\,\mathrm{d}x \\ &+ \int_0^T \int_{\mathbb{R}} \operatorname{sgn}(u(t,x)-v(t,x))[f(u(t,x))-f(v(t,x))]\partial_x\varphi \,\mathrm{d}t\,\mathrm{d}x \\ &+ \int_{\mathbb{R}} |u_0(x)-v_0(x)|\varphi(0,x)\,\mathrm{d}x \geqslant 0.\end{aligned}$$

(f) Define $M = \sup_{[-C,C]} |f'|$ with $C = \max(\|u\|_\infty, \|v\|_\infty)$ and a bounded interval $[a,b]$. For $t > 0$, by choosing an appropriate sequence of test functions φ_ε prove that for almost every $s \in [0,t]$

$$\int_{a-M(t-s)}^{b+M(t-s)} |u(s,x)-v(s,x)|\,\mathrm{d}x \leqslant \int_{a-Mt}^{b+Mt} |u_0(x)-v_0(x)|\,\mathrm{d}x.$$

(g) Deduce that $u_0 = v_0$ implies $v = u$ in $L^\infty(\mathbb{R}_+ \times \mathbb{R})$, and that $u_0 \geqslant 0$ almost everyhere on $\mathbb{R}$ implies $u \geqslant 0$ almost everywhere on $\mathbb{R}_+ \times \mathbb{R}$.

END OF PAPER