UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 7 June, 2017 9:00 am to 12:00 pm

PAPER 103

REPRESENTATION THEORY

Attempt no more than ***FOUR*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let $Z_n = Z(\mathbb{C}S_n)$. Let V be an irreducible $\mathbb{C}S_n$-module. Define the *Gelfand-Tzetlin basis* of V and the *Gelfand-Tzetlin subalgebra* GZ_n of $\mathbb{C}S_n$. Show that

(i) GZ_n is a maximal commutative subalgebra of $\mathbb{C}S_n$;

(ii) GZ_n is a semisimple algebra.

(b) Define the *Young-Jucys-Murphy elements* $X_1, X_2, \ldots, X_n$ and show directly that they commute.

Show that

(iii) GZ_n is generated by the subalgebras $Z_0, Z_1, \ldots, Z_n \subseteq \mathbb{C}S_n$;

(iv) GZ_n is generated by the Young-Jucys-Murphy elements $X_1, \ldots, X_n$.

[Olshanskii's lemma may be assumed.]

(c) Regarding S_{n-1} as the subgroup of S_n which acts on $\{1, 2, \ldots, n-1\}$, define a map $S_n \to S_{n-1}$ by sending $\pi \mapsto \pi_n$. Here, for $k = 1, 2, \ldots, n-1$, we set $\pi_n(k) = \pi(k)$, if $\pi(k) \neq n$ and $\pi_n(k) = \pi(n)$, if $\pi(k) = n$.

(v) Verify that (1) $(1_{S_n})_n = 1_{S_{n-1}}$, (2) $\sigma_n = \sigma$ for all $\sigma \in S_{n-1}$, and (3) $(\sigma\pi\theta)_n = \sigma\pi_n\theta$ for all $\pi \in S_n$ and $\sigma, \theta \in S_{n-1}$.

(vi) Suppose that $n \geqslant 4$. Let $\Phi : S_n \to S_{n-1}$ be a map satisfying the condition: $\Phi(\sigma\pi\theta) = \sigma\Phi(\pi)\theta$ for all $\pi \in S_n$ and $\sigma, \theta \in S_{n-1}$. Show that Φ necessarily coincides with the map defined in (c).

(vii) Deduce the following characterisation of X_n. Extend the map $\pi \mapsto \pi_n$ linearly to a map $\Pi_n : \mathbb{C}S_n \to \mathbb{C}S_{n-1}$. Show that, in the usual notation,

$$\Pi_n^{-1}(\langle 1_{S_{n-1}} \rangle) \cap Z_{((n-1),1)} = \langle X_n, 1_{S_n} \rangle,$$

where the angle brackets denote the appropriate span.

2

(a) Given $n \in \mathbb{N}$, define the sets $\mathrm{Spec}(n)$ and $\mathrm{Cont}(n)$.

(b) If $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathrm{Spec}(n)$, and assuming appropriate results, show that

(i) $\lambda_1 = 0$;

(ii) $\{\lambda_i - 1, \lambda_i + 1\} \cap \{\lambda_1, \dots, \lambda_{i-1}\} \neq \emptyset$ for all $1 < i \leqslant n$;

(iii) If $\lambda_i = \lambda_j = a$ for some $i < j$ then

$$\{a-1, a+1\} \subseteq \{\lambda_{i+1}, \dots, \lambda_{j-1}\}.$$

[Appropriate results may be assumed if clearly stated.]

(c) Show that $\mathrm{Cont}(n)$ is precisely the set of all n-tuples $\lambda \in \mathbb{C}^n$ which satisfy the properties (i)–(iii) listed above.

(d) Explain briefly how one can deduce the Branching Rule from the previous results (proofs are not required).

3

(a) State and prove Young's seminormal form regarding a certain choice of Gelfand-Tzetlin basis.

(b) Show that $V^{(n-1,1)}$ is isomorphic to the representation $\{(z_1, \dots, z_n) \in \mathbb{C}^n : \sum z_i = 0\}$ with S_n acting by place permutations.

Write down all the standard $(n-1,1)$–tableaux and their corresponding contents. Compute Young's orthogonal form for the action of the normalised Young basis vectors $\{w_T\}$ corresponding to each standard tableau.

(c) Recall that a partition (or Young diagram) λ is called a *hook* if it is of the form $\lambda = (n-k, 1^k)$ for some $0 \leqslant k \leqslant n-1$.

(i) Prove that the product $X_2 X_3 ... X_n$ of YJM elements equals the sum of all n–cycles of S_n.

(ii) If λ is not a hook and T is a standard λ-tableau, show that $X_2 \dots X_n w_T = 0$.

(iii) If λ is a hook of the above form and T is a standard λ-tableau, show that $X_2 \dots X_n w_T$ is a non-zero multiple of w_T, determining the multiple explicitly as a function of k and n.

(iv) If $\lambda \in \mathcal{P}(n)$, deduce an expression for $X_2 \dots X_n w$ for any $w \in V^\lambda$ and deduce an expression for the character χ^λ of V^λ evaluated at any element lying in the conjugacy class of cycle type (n). [You may not appeal directly to the Murnaghan-Nakayama Rule.]

4

Let $0 < k < n$.

(a) For $\lambda \in \mathcal{P}(n)$ and $\mu \in \mathcal{P}(n-k)$, and $\mu \preccurlyeq \lambda$, define a *skew shape* λ/μ and define what it means for a skew shape to be a *skew hook.* What does it mean for a skew shape λ/μ to be (i)*connected* or (ii) *totally disconnected*?

Let T be a standard λ/μ–tableau. Recall that a Coxeter transposition s_j is said to be *admissible* for T if s_jT is still standard. Let T, R be standard λ/μ–tableaux, $\pi \in S_k$ and suppose that $\pi T = R$. Show that R may be obtained from T by a sequence of $\ell(\pi)$ admissible transpositions.

(b) Define the *skew representation* $V^{\lambda/\mu}$ and explain why it has the structure of a $\mathbb{C}S_k$-module. For every standard λ/μ-tableau T, show that w_T is a cyclic vector in $V^{\lambda/\mu}$. [You may assume the existence of an orthonormal basis $\{w_T\}$ as T runs through the standard λ/μ-tableaux.]

(c) (i) Let (ρ, V) be a unitary representation of a finite group G. Let $V^G = \{u \in V : \rho(g)u = u \text{ for all } g \in G\}$, the subspace of all ρ-invariant vectors. Suppose that there exist a cyclic vector $v \in V$, $g \in G$ and $\lambda \in \mathbb{C}$, $\lambda \neq 1$, such that $\rho(g)v = \lambda v$. Prove that $V^G = \{0\}$.

(ii) Deduce that the multiplicity of the trivial representation $V^{(k)}$ in $V^{\lambda/\mu}$ is equal to 1 if λ/μ is totally disconnected and 0 otherwise.

5

(a) Define the *dominance ordering*, $\trianglerighteq$, on partitions of n.

Let $\lambda, \mu \in \mathcal{P}(n)$. We will say that μ is obtained from λ by a *single-box up-move* if there exist positive integers i and j with $i < j$ such that $\mu_\ell = \lambda_\ell$ for all $\ell \neq i, j$ and $\mu_i = \lambda_i + 1$ and $\lambda_j = \mu_j + 1$.

(b) Let $\lambda, \mu \in \mathcal{P}(n)$. Show that the following two conditions are equivalent:

(i) $\lambda \trianglelefteq \mu$.

(ii) There exists a chain

$$\lambda^0 \trianglelefteq \lambda^1 \trianglelefteq \cdots \trianglelefteq \lambda^{s-1} \trianglelefteq \lambda^s,$$

where $\lambda^0 = \lambda, \lambda^s = \mu$ and λ^{i+1} is obtained from λ^i by a single-box up-move, for $i = 0, 1, \ldots, s-1$.

(c) Prove that if $\lambda \trianglelefteq \mu$ then there exists an S_n-invariant subspace $L_{\lambda,\mu}$ in the permutation module M^λ such that $M^\lambda \cong M^\mu \oplus L_{\lambda,\mu}$. [*Hint: you may wish to recall that $M^{(n-k,k)}$ decomposes, without multiplicities, into the direct sum of $k+1$ irreducible representations.*]

6

(a) Explain briefly how to identify $V^{(1^n)}$ with the sign representation, `sgn`, of S_n.

(b) Define the *conjugate partition* λ' of λ. Show that for every partition λ of n,

$$V^{\lambda'} \cong V^{\lambda} \otimes V^{(1^n)}.$$

[Young's orthogonal form may be assumed.]

(c) With the usual notation for M^λ and $\tilde{M}^\lambda = \mathrm{Ind}_{S_\lambda}^{S_n}\mathrm{sgn}$, show that $\tilde{M}^{\lambda'} \cong V^{(1^n)} \otimes M^\lambda$. Calculate the inner product of the characters of M^λ and $\tilde{M}^{\lambda'}$.

(d) Let $M(\mu, \lambda)$ be the multiplicity of V^μ in M^λ. Deduce from (c) that $M(\lambda, \lambda) = 1$. [If you wish, you may assume without proof that $\dim \mathrm{Hom}_{S_n}(\tilde{M}^{\lambda'}, M^\lambda) = 1$.]

(e) State and prove Vershik's linear relations for the $M(\mu, \lambda)$.

END OF PAPER