MATHEMATICAL TRIPOS Part III

Wednesday, 7 June, 2017 9:00 am to 12:00 pm

PAPER 103

REPRESENTATION THEORY

Attempt no more than **FOUR** questions.

There are **SIX** questions in total.

The questions carry equal weight.

**STATIONERY REQUIREMENTS**

Cover sheet
Treasury Tag
Script paper

**SPECIAL REQUIREMENTS**

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.
(a) Let \( Z_n = \mathbb{Z}(\mathbb{C}S_n) \). Let \( V \) be an irreducible \( \mathbb{C}S_n \)-module. Define the Gelfand-Tzetlin basis of \( V \) and the Gelfand-Tzetlin subalgebra \( GZ_n \) of \( \mathbb{C}S_n \). Show that

(i) \( GZ_n \) is a maximal commutative subalgebra of \( \mathbb{C}S_n \);
(ii) \( GZ_n \) is a semisimple algebra.

(b) Define the Young-Jucys-Murphy elements \( X_1, X_2, \ldots, X_n \) and show directly that they commute.

(c) Regarding \( S_{n-1} \) as the subgroup of \( S_n \) which acts on \( \{1, 2, \ldots, n-1\} \), define a map \( S_n \rightarrow S_{n-1} \) by sending \( \pi \mapsto \pi_n \). Here, for \( k = 1, 2, \ldots, n-1 \), we set \( \pi_n(k) = \pi(k) \), if \( \pi(k) \neq n \) and \( \pi_n(k) = \pi(n) \), if \( \pi(k) = n \).

(v) Verify that (1) \( (1_{S_n})_n = 1_{S_{n-1}} \), (2) \( \sigma_n = \sigma \) for all \( \sigma \in S_{n-1} \), and (3) \( (\sigma\pi\theta)_n = \sigma\pi_n\theta \) for all \( \pi \in S_n \) and \( \sigma, \theta \in S_{n-1} \).

(vi) Suppose that \( n \geq 4 \). Let \( \Phi : S_n \rightarrow S_{n-1} \) be a map satisfying the condition: \( \Phi(\sigma\pi\theta) = \sigma\Phi(\pi)\theta \) for all \( \pi \in S_n \) and \( \sigma, \theta \in S_{n-1} \). Show that \( \Phi \) necessarily coincides with the map defined in (c).

(vii) Deduce the following characterisation of \( X_n \). Extend the map \( \pi \mapsto \pi_n \) linearly to a map \( \Pi_n : \mathbb{C}S_n \rightarrow \mathbb{C}S_{n-1} \). Show that, in the usual notation, 

\[
\Pi_n^{-1}(\langle 1_{S_{n-1}} \rangle) \cap Z_{(n-1),1} = \langle X_n, 1_{S_n} \rangle,
\]

where the angle brackets denote the appropriate span.
2

(a) Given \( n \in \mathbb{N} \), define the sets \( \text{Spec}(n) \) and \( \text{Cont}(n) \).

(b) If \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \text{Spec}(n) \), and assuming appropriate results, show that

(i) \( \lambda_1 = 0 \);

(ii) \( \{\lambda_i - 1, \lambda_i + 1\} \cap \{\lambda_1, \ldots, \lambda_{i-1}\} \neq \emptyset \) for all \( 1 < i \leq n \);

(iii) If \( \lambda_i = \lambda_j = a \) for some \( i < j \) then

\[
\{a - 1, a + 1\} \subseteq \{\lambda_{i+1}, \ldots, \lambda_{j-1}\}.
\]

[Appropriate results may be assumed if clearly stated.]

(c) Show that \( \text{Cont}(n) \) is precisely the set of all \( n \)-tuples \( \lambda \in \mathbb{C}^n \) which satisfy the properties (i)–(iii) listed above.

(d) Explain briefly how one can deduce the Branching Rule from the previous results (proofs are not required).

3

(a) State and prove Young’s seminormal form regarding a certain choice of Gelfand-Tzetlin basis.

(b) Show that \( V^{(n-1,1)} \) is isomorphic to the representation \( \{ (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum z_i = 0 \} \) with \( S_n \) acting by place permutations.

Write down all the standard \( (n-1,1) \)-tableaux and their corresponding contents. Compute Young’s orthogonal form for the action of the normalised Young basis vectors \( \{w_T\} \) corresponding to each standard tableau.

(c) Recall that a partition (or Young diagram) \( \lambda \) is called a hook if it is of the form \( \lambda = (n-k, 1^k) \) for some \( 0 \leq k \leq n-1 \).

(i) Prove that the product \( X_2 X_3 \ldots X_n \) of YJM elements equals the sum of all \( n \)-cycles of \( S_n \).

(ii) If \( \lambda \) is not a hook and \( T \) is a standard \( \lambda \)-tableau, show that \( X_2 \ldots X_n w_T = 0 \).

(iii) If \( \lambda \) is a hook of the above form and \( T \) is a standard \( \lambda \)-tableau, show that \( X_2 \ldots X_n w_T \) is a non-zero multiple of \( w_T \), determining the multiple explicitly as a function of \( k \) and \( n \).

(iv) If \( \lambda \in \mathcal{P}(n) \), deduce an expression for \( X_2 \ldots X_n w \) for any \( w \in V^\lambda \) and deduce an expression for the character \( \chi^\lambda \) of \( V^\lambda \) evaluated at any element lying in the conjugacy class of cycle type \( (n) \). [You may not appeal directly to the Murnaghan-Nakayama Rule.]
Let $0 < k < n$.

(a) For $\lambda \in \mathcal{P}(n)$ and $\mu \in \mathcal{P}(n-k)$, and $\mu \preceq \lambda$, define a skew shape $\lambda/\mu$ and define what it means for a skew shape to be a skew hook. What does it mean for a skew shape $\lambda/\mu$ to be (i) connected or (ii) totally disconnected?

Let $T$ be a standard $\lambda/\mu$-tableau. Recall that a Coxeter transposition $s_j$ is said to be admissible for $T$ if $s_j T$ is still standard. Let $T, R$ be standard $\lambda/\mu$-tableaux, $\pi \in S_k$ and suppose that $\pi T = R$. Show that $R$ may be obtained from $T$ by a sequence of $\ell(\pi)$ admissible transpositions.

(b) Define the skew representation $V^{\lambda/\mu}$ and explain why it has the structure of a $\mathbb{C}S_k$-module. For every standard $\lambda/\mu$-tableau $T$, show that $w_T$ is a cyclic vector in $V^{\lambda/\mu}$. [You may assume the existence of an orthonormal basis $\{w_T\}$ as $T$ runs through the standard $\lambda/\mu$-tableaux.]

(c) (i) Let $(\rho, V)$ be a unitary representation of a finite group $G$. Let $V^G = \{u \in V : \rho(g)u = u \text{ for all } g \in G\}$, the subspace of all $\rho$-invariant vectors. Suppose that there exist a cyclic vector $v \in V$, $g \in G$ and $\lambda \in \mathbb{C}$, $\lambda \neq 1$, such that $\rho(g)v = \lambda v$. Prove that $V^G = \{0\}$.

(ii) Deduce that the multiplicity of the trivial representation $V^{(k)}$ in $V^{\lambda/\mu}$ is equal to 1 if $\lambda/\mu$ is totally disconnected and 0 otherwise.

5

(a) Define the dominance ordering, $\succeq$, on partitions of $n$.

Let $\lambda, \mu \in \mathcal{P}(n)$. We will say that $\mu$ is obtained from $\lambda$ by a single-box up-move if there exist positive integers $i$ and $j$ with $i < j$ such that $\mu_\ell = \lambda_\ell$ for all $\ell \neq i, j$ and $\mu_i = \lambda_i + 1$ and $\mu_j = \lambda_j + 1$.

(b) Let $\lambda, \mu \in \mathcal{P}(n)$. Show that the following two conditions are equivalent:

(i) $\lambda \preceq \mu$.

(ii) There exists a chain $\lambda^0 \preceq \lambda^1 \preceq \cdots \preceq \lambda^{s-1} \preceq \lambda^s$, where $\lambda^0 = \lambda, \lambda^s = \mu$ and $\lambda^{i+1}$ is obtained from $\lambda^i$ by a single-box up-move, for $i = 0, 1, \ldots, s-1$.

(c) Prove that if $\lambda \preceq \mu$ then there exists an $S_n$-invariant subspace $L_{\lambda, \mu}$ in the permutation module $M^\lambda$ such that $M^\lambda \cong M^\mu \oplus L_{\lambda, \mu}$. [Hint: you may wish to recall that $M^{(n-k,k)}$ decomposes, without multiplicities, into the direct sum of $k+1$ irreducible representations.]
(a) Explain briefly how to identify $V^{(1^n)}$ with the sign representation, $\text{sgn}$, of $S_n$.

(b) Define the conjugate partition $\lambda'$ of $\lambda$. Show that for every partition $\lambda$ of $n$,

$$V^{\lambda'} \cong V^\lambda \otimes V^{(1^n)}.$$  

[Young’s orthogonal form may be assumed.]

(c) With the usual notation for $M^\lambda$ and $\tilde{M}^\lambda = \text{Ind}_{S_\lambda}^{S_n} \text{sgn}$, show that $\tilde{M}^{\lambda'} \cong V^{(1^n)} \otimes M^\lambda$. Calculate the inner product of the characters of $M^\lambda$ and $\tilde{M}^{\lambda'}$.

(d) Let $M(\mu, \lambda)$ be the multiplicity of $V^\mu$ in $M^\lambda$. Deduce from (c) that $M(\lambda, \lambda) = 1$. [If you wish, you may assume without proof that $\dim \text{Hom}_{S_n}(\tilde{M}^{\lambda'}, M^\lambda) = 1$.]

(e) State and prove Vershik’s linear relations for the $M(\mu, \lambda)$.

END OF PAPER