UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 5 June, 2017 9:00 am to 12:00 pm

PAPER 102

LIE ALGEBRAS AND THEIR REPRESENTATIONS

*Attempt all **FIVE** questions.*

Questions 1, 2 and 5 carry 20 marks.

Question 3 carries 30 marks.

Question 4 carries 10 marks.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Define what it means for a non-abelian Lie algebra $\mathfrak{g}$ to be *simple*.

Let $\mathfrak{g} = \mathfrak{sl}_n$, the Lie algebra of trace zero $n \times n$-matrices over a field k.

i) Show that $\mathfrak{g}$ is simple if $k = \mathbb{C}$.

ii) Suppose that k is algebraically closed, and of characteristic p.

Show $\mathfrak{g} = \mathfrak{sl}_p$ is *not* simple.

Show that if $p > 2$, the quotient of $\mathfrak{g}$ by the center of $\mathfrak{g}$ is simple. [You need not write out the proof in detail – just indicate how, if at all, it differs from what you wrote in (i).] Show that if $p = 2$ this quotient is abelian.

2

i) Let $\mathfrak{g}$ be a Lie algebra over a field k, and V be a finite dimensional representation of $\mathfrak{g}$. Define what it means for a bilinear form $(\,,\,) : V \times V \to k$ to be *invariant*.

Now suppose that V is an irreducible module for $\mathfrak{g}$, and k is algebraically closed. Show that any two such forms $(\,,\,)_1$, $(\,,\,)_2$ are proportional, that is if $(\,,\,)_1 \neq 0$, there exists a $\lambda \in k$ with $(\,,\,)_2 = \lambda(\,,\,)_1$.

Show that, if *char* $k \neq 2$, any such form is either symmetric or anti-symmetric.

ii) Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Show that *every* finite dimensional representation of $\mathfrak{g}$ admits a non-degenerate invariant bilinear form.

For each irreducible finite dimensional representation of $\mathfrak{sl}_2$, describe whether the bilinear form on it is symmetric or anti-symmetric.

3

Let
$$\mathfrak{g} = \mathfrak{so}_{2n+1}(\mathbb{C}) = \left\{A \in Mat_{2n+1}(\mathbb{C}) | AJ + JA^T = 0\right\}$$

where $J = \begin{pmatrix} & & & & & 1 \\ & & & & \ddots & \\ & & & 1 & & \\ & & 1 & & & \\ & \ddots & & & & \\ 1 & & & & & \end{pmatrix}$, and let $\mathfrak{t}$ = diagonal matrices in $\mathfrak{g}$.

i) Decompose $\mathfrak{g}$ as a $\mathfrak{t}$-module, and hence write the roots R for $\mathfrak{g}$. Choose positive roots to be those occurring in upper triangular matrices. Write down the positive roots R^+, the simple roots Π, the highest root θ, and the fundamental weights.

Write down ρ.

Draw the Dynkin diagram and label it by simple roots. Draw the extended Dynkin diagram.

ii) Show that $\mathfrak{so}_5 \simeq \mathfrak{sp}_4$.

iii) For each root $\alpha \in R$, write the reflection $s_\alpha : \mathfrak{t} \to \mathfrak{t}$ explicitly. Describe the Weyl group W (you do not need to prove your answer).

iv) Let $V = \mathbb{C}^{2n+1}$ be the standard representation of $\mathfrak{so}_{2n+1}$. Draw the crystal of V, and of $V \otimes V$. Write the highest weight of each irreducible summand of $V \otimes V$.

4

(i) State the Weyl character formula, and the Weyl denominator formula,briefly defining the notation you use.

(ii) Show that $ch\, L(k\rho) = e^{k\rho} \prod_{\alpha \in R^+} (1 + e^{-\alpha} + \cdots + e^{-k\alpha})$.

5

Draw the root system of G_2.

(i) Write down the dimension of the irreducible representation with highest weight $n_1\omega_1 + n_2\omega_2$, when $n_1, n_2 \in \mathbb{N}$.

(ii) Let V be the 7 dimensional representation. What is its highest weight? Draw the crystal of V.

(iii) Decompose $\bigwedge^2 V$ and S^2V into irreducibles, when V is the 7 dimensional representation. [You may use any results from the course, just as long as they are clearly stated].

END OF PAPER