UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 6 June, 2016 1:30 pm to 3:30 pm

PAPER 336

PERTURBATION METHODS

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

a) Obtain, up to and including terms of $O(\varepsilon)$, a perturbation series for

$$I(\varepsilon) = \int_{\varepsilon}^{\infty} \frac{e^{-x} e^{-\varepsilon/x}}{x} \, \mathrm{d}x$$

in the limit $\varepsilon \to 0^+$, expressing your answer in terms of Euler's constant γ and the constant $\beta \equiv \int_1^\infty t^{-1} e^{-t} \, \mathrm{d}t$. You may use the perturbation series for the exponential integral $E(z)$:

$$E(z) = \int_z^\infty t^{-1} e^{-t} \, \mathrm{d}t \sim -\log z - \gamma + z \qquad \text{as } z \to 0^+.$$

b) The function $f(\theta, \lambda)$ is defined by

$$f(\theta, \lambda) = \int_{-\frac{1}{\sqrt{2}}}^{e^{i\theta}} e^{-\lambda z^2} \, \mathrm{d}z,$$

where θ and λ are real, $0 \leqslant \theta \leqslant \pi$, and the integration contour is the straight-line segment of the complex plane connecting the limits of the integral.

i) Show that, for $\lambda \to \infty$,

$$f(0, \lambda) \sim \sqrt{\frac{\pi}{\lambda}} \qquad \text{and} \qquad f(\pi, \lambda) \sim -\frac{1}{\sqrt{2}\,\lambda} e^{-\frac{1}{2}\lambda}.$$

[You may assume that $\int_{-\infty}^{\infty} e^{-x^2} \, \mathrm{d}x = \sqrt{\pi}$.]

ii) Use the method of steepest descents to determine all the leading-order asymptotic behaviours of $f(\theta, \lambda)$ in the limit $\lambda \to \infty$ for $0 \leqslant \theta \leqslant \pi$, and the values of θ for which each applies. Your answer should include sketches of the chosen steepest descent paths.

2

(a) The function $y(x)$ satisfies the differential equation

$$\varepsilon y'' + 2xy' - 2xy = 0$$

and boundary conditions

$$y(0) = 0\,, \quad y(1) = e\,,$$

where $0 < \varepsilon \ll 1$. Using matched asymptotic expansions, find the solution for $y(x)$ correct to and including $O(\varepsilon)$ terms, in both inner and outer regions, for $0 \leqslant x \leqslant 1$.

[*Hints:*

(i) *Recall that*

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\,\mathrm{d}t \qquad \textit{and} \qquad \mathrm{erf}(\infty) = 1\,.$$

(ii) A particular solution for $Y(z)$ to

$$Y'' + 2zY' = 2a_1 z\mathrm{erf}(z) + 2a_2 z + 2a_3 z e^{-z^2}\,,$$

where a_1, a_2 and a_3 are constants, is

$$Y = a_1 \left(\frac{2}{\sqrt{\pi}} e^{-z^2} + z\mathrm{erf}(z) \right) + a_2 z - \tfrac{1}{2} a_3 z e^{-z^2}\,.$$

(iii) A particular solution for $Y(z)$ to

$$Y'' + 2zY' = 2z^2\mathrm{erf}(z)$$

is

$$Y = \tfrac{1}{2} z^2 \mathrm{erf}(z) + \frac{1}{\sqrt{\pi}} z e^{-z^2} - \int_0^z e^{-t^2} \int_0^t e^{u^2} \mathrm{erf}(u)\,\mathrm{d}u\,\mathrm{d}t\,,$$

where

$$\int_0^z e^{-t^2} \int_0^t e^{u^2} \mathrm{erf}(u)\,\mathrm{d}u\,\mathrm{d}t \sim \tfrac{1}{2}\log z + C \qquad \textit{as } z \to \infty$$

and C can be taken as a known constant.]

(b) The function $w(x)$ satisfies the differential equation

$$\varepsilon x w'' + w' + 2xw = 0\,,$$

and boundary conditions

$$w(\varepsilon) = 0\,, \quad w(1) = e^{-1}\,,$$

where $0 < \varepsilon \ll 1$. Using matched asymptotic expansions, find the solution for $w(x)$ correct to and including $O(\varepsilon)$ terms, in both inner and outer regions, for $\varepsilon \leqslant x \leqslant 1$.

[*Hints: A particular solution to*

$$w' + 2xw = 2x(1-2x^2)e^{-x^2} \qquad \textit{is} \qquad w = x^2(1-x^2)e^{-x^2}\,,$$

and a particular solution to

$$w'' + w' = 2xe^{-x} \qquad \textit{is} \qquad w = -x(2+x)e^{-x}\,.]$$

3

Waves propagating through a slowly-varying medium satisfy the wave equation

$$\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial}{\partial x}\left(c^2\frac{\partial\varphi}{\partial x}\right) - \frac{\partial}{\partial y}\left(c^2\frac{\partial\varphi}{\partial y}\right) = 0\,, \tag{1}$$

where the wavespeed c is a slowly varying function of $\mathbf{x} = (x, y)$ and t, i.e.

$$c \equiv c(\mathbf{X}, T)\,,$$

where $\mathbf{X} = (X, Y) = \varepsilon\mathbf{x}$, $T = \varepsilon t$ and $0 < \varepsilon \ll 1$. On the assumption that

$$\varphi = (A_0(\mathbf{X}, T) + \varepsilon A_1(\mathbf{X}, T) + \dots)\exp\left(\frac{i}{\varepsilon}\theta(\mathbf{X}, T)\right) + \text{c.c.}\,,$$

derive the ray equations for waves with order-one local frequencies, ω, and wavenumbers, $\mathbf{k} = (k, \ell)$, where ω and $\mathbf{k}$ should be defined. Show that

$$(\omega A_0^2)_T + (kc^2A_0^2)_X + (\ell c^2 A_0^2)_Y = 0\,,$$

and that

$$\frac{\partial k}{\partial Y} = \frac{\partial l}{\partial X}\,.$$

Show also that on a ray specified by

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}T} = \mathbf{c}_g \equiv \left(\frac{\partial\Omega}{\partial k}, \frac{\partial\Omega}{\partial\ell}\right)\,,$$

where the local dispersion relation is given by $\omega = \Omega(\mathbf{k}; \mathbf{X}, T)$,

$$\frac{\mathrm{d}\mathbf{k}}{\mathrm{d}T} = -\left(\frac{\partial\Omega}{\partial X}, \frac{\partial\Omega}{\partial Y}\right)\,, \quad \frac{\mathrm{d}\omega}{\mathrm{d}T} = \frac{\partial\Omega}{\partial T}\,.$$

Suppose henceforth that the wavespeed c is independent of X and T, i.e. that $c \equiv c(Y)$, and that $\frac{\mathrm{d}c}{\mathrm{d}Y} > 0$. Suppose also that the amplitude A_0 is independent of X and T. Assuming that $\ell > 0$, show that

$$A_0 = \frac{\mu}{c\sqrt{\ell}}\,,$$

for some constant μ.

Next, suppose that there exists Y_s such that $c(Y_s) = \omega/k$, and suppose that a wavefield exists in $Y < Y_s$. Comment on the validity of ray theory as $Y \to Y_s-$. Deduce an inner scaling $(Y - Y_s) = \delta\eta$, where $\eta = O(1)$ and $\delta \equiv \delta(\varepsilon) \ll 1$ is to be determined, and derive from equation (1) the form of the leading-order solution for φ. Assuming that $\mu = O(1)$, identify the order-of-magnitude of φ in the inner region; there is no need to formally match the coefficients in the inner and outer solutions.

[*Hint: The solution for* $\psi(z)$ *of*

$$\psi_{zz} - z\psi = 0$$

that decays as $z \to \infty$ *is* $\mathrm{Ai}(z)$, *where*

$$\mathrm{Ai}(z) \to \frac{1}{\sqrt{\pi}(-z)^{\frac{1}{4}}}\sin\left(\tfrac{2}{3}(-z)^{\frac{3}{2}} + \tfrac{\pi}{4}\right) \qquad \text{as } z \to -\infty.]$$

END OF PAPER