UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 3 June, 2016 9:00 am to 11:00 am

PAPER 335

DIRECT AND INVERSE SCATTERING OF WAVES

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider a time-harmonic plane wave $\psi(x,z)$ with wave number k travelling in 2-dimensional free space at a small angle α to the horizontal x-axis.

(a) Derive the parabolic equation for the reduced wave

$$E(x,z) = \psi(x,z)e^{-ikx} \ , \tag{1}$$

and write the solution $E(x,z)$ in terms of an initial value $E(0,z)$.

(b) Now consider the acoustic field $\psi(x,z)$ generated by an incident time-harmonic wave $\psi_i(x,z) = e^{ikx}$ when the 2-dimensional free space includes an inhomogeneity with refractive index $n(z)$, confined to a thin vertical layer in a region $x \in [-\xi, 0]$. We assume that the layer is weakly scattering, with randomly varying refractive index given by

$$n(z) = 1 + \mu W(z) \ , \qquad \text{with } \mu \ll 1 \ , \tag{2}$$

where W is normally distributed and statistically stationary, with mean and variance given by $< W >= 0$ and $< W^2 >= 1$.

Assume that the thin layer only imposes a phase shift $\phi(z) = k\xi\mu W(z)$ on the wave. Consider the Fourier transform w.r.t. z of the reduced field:

$$\hat{E}(x,\nu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(x,z)e^{-i\nu z}\, dz. \tag{3}$$

Write the ensemble average of the time-averaged energy flux of $\hat{E}(x,\nu)$ at $x > 0$ in the direction of the x-axis.

Then express this ensemble average in terms of the autocorrelation function of the layer $\rho(\zeta)$.

(c) For the same time-harmonic wave $\psi_i(x,z) = e^{ikx}$ incident on the thin vertical layer defined in (b), calculate the ensemble average of the Fourier transform w.r.t. z of the reduced field $\hat{E}(x,\nu)$ at $x > 0$.

Comment on its dependence on x, and relate this to the dependence on x of the ensemble average of the time-averaged energy flux of $\hat{E}(x,\nu)$ calculated in (b).

2

Consider the 3-dimensional wave equation in the frequency domain, for an acoustic field $\psi(\mathbf{r}, k)$ with a source $Q(\mathbf{r})$:

$$\nabla^2 \psi(\mathbf{r}, k) + k^2 \psi(\mathbf{r}, k) = Q(\mathbf{r}) \ , \qquad (1)$$

where we assume the source $Q(\mathbf{r})$ is a square-integrable function supported on the finite region A, defined as a sphere of radius $\mathbf{r}_0$ centred at the origin, so $Q(\mathbf{r}) \in L^2(A)$.

(a) Define the free-space Green's function for this problem, and show how it can be used to write the solution $\psi(\mathbf{r}, k)$ of equation (1).

(b) Derive a far-field approximation for the solution $\psi(\mathbf{r}, k)$ obtained in (a), and use it to derive an integral expression for the far field pattern in this problem, using an explicit expression of the Green's function.

Hence, define the operator $T : L^2(A) \mapsto L^2(S_1)$ which maps a source $Q \in L^2(A)$ onto a far field pattern $f_\infty(\hat{\mathbf{r}}, k) \in L^2(S_1)$, where S_1 is the surface of the unit sphere, and $\hat{\mathbf{r}}$ is the unit vector in the direction $\mathbf{r}$.

Find an expression for the adjoint of T, T^*.

(c) Consider now the inverse problem of finding the source $Q(\mathbf{r})$, given the far field pattern $f_\infty(\hat{r}, k)$.

Formulate this problem using the normal equation, and hence write a formal expression for the solution $Q(\mathbf{r})$.

Comment on possible sources of ill-posedness for this particular problem.

3

Given the equation

$$Ax = y \ , \tag{1}$$

where $A : X \mapsto Y$ is a given compact linear operator between two Hilbert spaces, and $x \in X, \ y \in Y$, consider the inverse problem of finding x, given data y.

(a) Define a *regularisation strategy* for this inverse problem, then define *Tikhonov regularisation* and write a formal expression for the *Tikhonov regularised solution* of (1).

Explain why the Tikhonov regularised solution is stable.

(b) Define a *singular value system* $\{\sigma_i; u_i, v_i\}$ for A, and show how it can be used to construct the Tikhonov regularised solution.

(c) The iterated Tikhonov method for (1) is defined by:

$$\alpha x_{n+1} = \alpha x_n - A^* A x_n + A^* y \tag{2}$$

for some $\alpha > 0$, and with the first term x_0 given by the Tikhonov regularised solution derived in (b). Note that, similarly to Landweber iteration, the $(n+1)^{th}$ term in iterated Tikhonov can be written in closed form, i.e. as a function of $A^* y$ only, and not of x_n.

By using the singular value system and the closed form of (2), show that the n^{th} iterate in the iterated Tikhonov method can be written as

$$x_n = \sum_{i=1}^{\infty} g_\alpha (y, u_i) v_i \ , \tag{3}$$

where $(\cdot, \cdot)$ denotes an inner product.

Derive an expression for the function g_α.

[Hint: you may wish to use the partial sum formula $\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}$.]

END OF PAPER