UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 31 May, 2016 1:30 pm to 4:30 pm *Draft 22 June, 2016*

PAPER 334

ACTIVE BIOLOGICAL FLUIDS

*Attempt **ALL** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Two rigid spheres of radii a_1 and a_2 are located in an infinite Newtonian fluid. All inertial effects are neglected. The distance between the centres of the spheres is denoted by $\ell(t)$ and satisfies $\ell \gg a_1, a_2$.

A mechanical link between the spheres of variable length and negligible hydrodynamic resistance is used to vary $\ell(t)$ in a prescribed, time-periodic fashion. Including the leading-order hydrodynamic interactions between the spheres, show explicitly that the two-sphere system cannot move on average if it is force-free. Interpret your result in the context of the scallop theorem.

External forces are now used to move the two spheres independently in a periodic fashion as $x_1(t) = \delta \cos \omega t$ and $x_2(t) = \ell_0 + \delta \cos(\omega t + \phi)$, where x_i denotes the position along the x-axis of the centre of sphere i, and ω, δ and ℓ_0 are constants, such that $\ell_0 \geqslant \delta$. Denote by F_1 the force applied by sphere 1 on the fluid. Calculate the leading-order value of the time-averaged force, $\langle F_1 \rangle$, in powers of $1/\ell_0$. Deduce that, except for specific values of ϕ, a net force is induced on the fluid. Are your results in contradiction with the scallop theorem?

[*You may quote without proof the drag on an isolated sphere and the velocity field of a stokeslet*]

2

An infinite two-dimensional waving sheet is swimming under a viscous fluid using two travelling-wave modes operating at different frequencies and wavelengths. At small amplitude in the waving motion, the dimensionless location of material points (x_s, y_s) in the frame swimming with the sheet is given by

$$x_s = x + \epsilon a \sin 3(x-t), \quad y_s = \epsilon b \sin(x-t),$$

where (x, y) denotes the swimming frame of reference and t is the dimensionless time. Here $a > 0$, $b > 0$ and $\epsilon \ll 1$. As a result of its waving motion, the sheet swims with steady speed $-U\mathbf{e}_x$ with respect to the fluid at rest at infinity. Inertial effects in the fluid, which occupies the region above the sheet, are neglected.

What are the equations and boundary conditions satisfied by the streamfunction ψ in the frame moving with the sheet?

Solving the problem as a perturbation expansion in powers of ϵ, i.e. $\psi = \epsilon\psi_1 + \epsilon^2\psi_2 + ...$, derive the equation and the boundary conditions satisfied by ψ_1. Find the solution for ψ_1.

Derive the equation and boundary conditions for ψ_2 and the general form of its solution. Show how this can be used to calculate the swimming speed at order ϵ^2 without requiring the full solution for ψ_2.

Show that the swimming speed is zero when the ratio b/a takes a specific value.

Hint: You may use without proving it that the general unit-speed 2π-periodic travelling-wave solution to $\nabla^4 f(x, y, t) = 0$ is

$$f(x,y,t) = A + By + Cy^2 + Dy^3 + \mathcal{R}\left\{\sum_n \left[E_n e^{-ny} + F_n e^{ny} + y(G_n e^{-ny} + H_n e^{ny})\right] e^{in(x-t)}\right\}$$

3

An active slender filament of length L undergoes prescribed planar deformation in a Newtonian fluid resulting in swimming. In a frame (x, y, z) moving with the swimmer, the material points on the filament are located at $(x_s = x,\ y_s = \epsilon g(x,t),\ z_s = 0)$, for $0 \leqslant x \leqslant L$, where $\epsilon \ll 1$ is a dimensionless parameter. The swimming frame (x, y, z) is defined relative to the filament by prescribing that the shape function g satisfies $g(0,t) = \dfrac{\partial g}{\partial x}(0,t) = 0$. The swimming motion (i.e. the motion of the (x, y, z) frame) is described by the instantaneous translational speed, $U\mathbf{e}_x + V\mathbf{e}_y$, and instantaneous angular velocity, $\Omega\mathbf{e}_z$, of the origin and orientation of the swimming frame measured with respect to the frame in which the fluid is at rest at infinity. Inertial effects in the fluid are neglected.

The distribution of hydrodynamic forces along the filament is described using resistive-force theory. State this theory and explain the basic assumptions behind it.

Determine the distribution of velocities of the filament relative to the background fluid, $\mathbf{u}(x,t)$, for all points $0 \leqslant x \leqslant L$.

Anticipating that the swimming kinematics are to be solved as a perturbation expansion, $\{U, V, \Omega\} = \epsilon\{U_1, V_1, \Omega_1\} + O(\epsilon^2)$, explain why it is only necessary to characterise the tangent vector along the filament to $O(1)$ in order to solve the problem at $O(\epsilon)$.

Compute the total (instantaneous) hydrodynamic forces $F_x(t)$ and $F_y(t)$ and moment $M_z(t)$ on the filament at $O(\epsilon)$. Deduce that, for free-swimming motion, $U_1 = 0$. Compute the instantaneous values of V_1 and Ω_1 as functions of $\left\langle \dfrac{\partial g}{\partial t} \right\rangle$ and $\left\langle x\dfrac{\partial g}{\partial t} \right\rangle$ where $\left\langle ... \right\rangle \equiv \dfrac{1}{L}\displaystyle\int_0^L ...\mathrm{d}x$.

If the function $g(x,t)$ is periodic in time, show that both V_1 and Ω_1 time-average to zero.

4

A left-handed rigid helix immersed in a viscous fluid translates and rotates with velocity U and angular velocity Ω along its axis $\mathbf{e}_z$. Inertial effects in the fluid are neglected. The total length of the helix is ℓ, the radius of the cylinder around which it is coiled is a, and $0 < \theta < \pi/2$ is the angle between the local tangent to the helix centreline and $\mathbf{e}_z$. Writing the linear relationship between the kinematics of the helix (U, Ω) and the viscous force F and moment L acting on the helix in the $\mathbf{e}_z$ direction as

$$\begin{pmatrix} F \\ L \end{pmatrix} = -\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} U \\ \Omega \end{pmatrix},$$

calculate analytically the values of A, B, C and D using resistive-force theory. Verify that $B = C$.

Consider now the case of two helices, H_1 and H_2: helix H_1 has the same shape as described above, and is of length ℓ; helix H_2 has the same radius a and tangent angle θ, but is right-handed and has length $n\ell$, where $n \geqslant 0$. H_1 and H_2 are aligned with their axes along the $\mathbf{e}_z$ axis and linked in such a way that they cannot undergo relative translation but they do undergo a prescribed relative rotation of magnitude $\omega\mathbf{e}_z$ induced by a rotary motor of negligible hydrodynamic influence. If the two-helix system is force- and torque-free, compute the rotational speed of each helix and the total translational speed of the two-helix system. Interpret physically the results for the special cases $n = 0$, $n = 1$ and the limit $n \to \infty$.

END OF PAPER