UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 1 June, 2016 1:30 pm to 4:30 pm *Draft 21 June, 2016*

PAPER 333

FLUID DYNAMICS OF CLIMATE

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The wind blows on the surface of the ocean and exerts horizontal stresses (X, Y) in the x and y directions, respectively . By consideration of the net force on a thin horizontal slab, show that the linearised horizontal momentum equations on an f-plane are

$$u_t - fv = -\frac{1}{\rho}p_x + \frac{1}{\rho}X_z,$$
$$v_t + fu = -\frac{1}{\rho}p_y + \frac{1}{\rho}Y_z,$$

where the usual notation is applied and subscript indicates partial differentiation.

The wind produces an Ekman layer near the surface where stresses are important and a lower region where they are not. Divide the flow into a pressure-driven part (u^P, v^P) outside the surface Ekman layer and a stress-driven part (u^E, v^E) inside the Ekman layer, and write down the momentum equations for both parts. By integrating across the Ekman layer show, for steady flow, that the vertical Ekman velocity w^E at the base of the Ekman layer is

$$w^E = -\frac{1}{\rho f}(Y^s_x - X^s_y),$$

where (X^s, Y^s) are the surface stresses.

In the case where the surface stress is transmitted by laminar viscosity ν, by considering $u^E + iv^E$, find the velocity profile in the Ekman layer and show that

$$X^s = \rho\sqrt{\frac{f\nu}{2}}(u^P - v^P), \; Y^s = \rho\sqrt{\frac{f\nu}{2}}(u^P + v^P),$$

and

$$w^E = \frac{1}{\rho f}\sqrt{\frac{\nu}{2f}}(p_{xx} + p_{yy}).$$

Now consider the ocean as a shallow layer of depth H and free surface elevation η. The forced, linearised shallow water equations below the Ekman layer are

$$u_t - fv = -g\eta_x, \; v_t + fu = -g\eta_y,$$
$$\eta_t + H(u_x + v_y) = -\eta^E_t,$$

where $w^E = \eta^E_t$.

This pumping causes the ocean to 'spin down' on a time scale τ. Assuming $f\tau \ll 1$ show that

$$(\eta_{xx} + \eta_{yy} - \frac{f^2}{c^2}\eta)_t = -\frac{1}{H}\sqrt{\frac{f\nu}{2}}(\eta_{xx} + \eta_{yy}),$$

where $c = \sqrt{gH}$. Find the spin down timescale τ for a sinusoidal disturbance of wavenumber κ and show that it is independent of κ for disturbances with scales small compared with the Rossby radius of deformation. In the case when the scales are large compared to the deformation scale show that disturbances decay with a diffusivity K^E

$$K^E = g\sqrt{\frac{f\nu}{2}}/f^2.$$

2

The linearised shallow water equations for a single layer of fluid of constant depth H on a f-plane reduce to a single equation for the free surface elevation η

$$\eta_{tt} + f^2\eta - c^2(\eta_{xx} + \eta_{yy}) \quad = \quad -H\mathbf{f}\cdot\mathbf{q},$$

where $\mathbf{q} = \nabla_h \times \mathbf{u} - \eta\mathbf{f}/H$, ∇_h is the horizontal gradient operator $c^2 = gH$ and $f = |\mathbf{f}|$. Give a physical interpretation of $\mathbf{q}$

Consider a layer of fluid initially at rest and for which at $t = 0$, $\eta(x, 0) = \eta_0 \operatorname{sgn}(x)$.

Show that this initial state adjusts to a final state given by

$$\eta = \eta_0 \begin{cases} 1 - e^{-x/R_D}, & x > 0, \\ -1 + e^{x/R_D}, & x < 0, \end{cases}$$

where $R_D = c/f$. Show also that the volume flux of the adjusted flow in the $y-$direction is $2c^2\eta_0/f$.

Now consider the case where the depth is discontinuous along the $x-$axis

$$H = \begin{cases} H^-, & y < 0, \\ H^+, & y > 0, \end{cases}$$

where $H^- < H^+$, so that the flux leaving the step $y > 0$ is greater than that arriving from the shallow side $y < 0$.

This difference is accommodated by a *double Kelvin wave* in which flow is directed along the step and surface elevation approaches η_0 as $y \to \pm\infty$ given by

$$\eta \quad = \quad \eta_0 \operatorname{sgn}(x) - A(x, t)e^{-|y|/R_D^{\pm}},$$

where $R_D^{\pm} = \sqrt{gH^{\pm}}f^{-1}$, and where $A(x, t)$ is to be determined.

Using geostrophy and the fact that Hv is continuous at $y = 0$, show that

$$A_t + \Delta c A_x \quad = \quad \Delta c\eta_0\delta(x), \qquad (1)$$

where $\Delta c = \sqrt{gH^+} - \sqrt{gH^-}$.

Find the solution of (1) subject to $A(x, 0) = 0$, and show that the step acts as a complete barrier to the flow.

3

The shallow water potential vorticity (PV) is

$$q = \frac{f + \zeta}{H}, \tag{1}$$

where f is the Coriolis parameter, ζ is the vertical component of the relative vorticity, and H is the fluid depth.

i) Describe the response of a parcel of fluid in the two following scenarios, providing a brief physical interpretation in each case:

  a) A vertical column of fluid moves into deeper water at the same latitude

  b) A vertical column of fluid moves towards the equator without changing its relative vorticity.

ii) Starting from the shallow water PV conservation equation, or otherwise, derive an expression for the quasi-geostrophic PV. Explicitly state all assumptions required.

iii) A depth-independent (barotropic) jet, $U(y)$, flows from the west to the east. Starting from the QG equations and invoking the *beta*-plane approximation, derive the dispersion relation for linear plane waves.

iv) A uniform wind, U, blows from the west to the east over an isolated mountain and generates waves whose phase is stationary with respect to the ground. Using your result from the previous part, find an expression for the wavelength and group velocity of these waves. Sketch the region containing all waves generated between times $0 < t < \tau$. What happens if the wind instead blows from the east to the west?

4

Potential vorticity conservation for forced shallow water flow in the ocean can be written

$$\frac{D}{Dt}\left(\frac{f+\zeta}{H}\right) = \frac{F}{H} - \frac{r}{H}\zeta, \qquad (1)$$

where D/Dt is the material derivative, ζ is the vertical component of the relative vorticity, f is the Coriolis parameter, H is the fluid depth, F represents forcing by a wind stress curl, and r is a decay rate associated with bottom stress.

i) Show that the depth-integrated velocity can be written in terms of a streamfunction, ψ, and obtain an expression relating the streamfunction to the vorticity, ζ.

ii) Consider steady circulation with small Rossby number, but do not assume that changes in the fluid depth, H are small relative to the total depth. Show that the streamfunction satisfies the following equation

$$\widetilde{\mathbf{u}} \cdot \nabla\psi = \frac{r}{H}\nabla^2\psi - F - r\frac{\nabla H \cdot \nabla\psi}{H^2}, \qquad (2)$$

and obtain an expression for the *pseudovelocity*, $\widetilde{\mathbf{u}}$. Find the general solution for ψ for the unforced problem with $r = F = 0$.

iii) A uniform depth fluid is forced by a sinusoidal wind stress curl, $F = sin\,(2\pi y/L_y)$ in the domain $0 \leqslant y \leqslant L_y$. If the bottom stress is relatively weak compared with the wind stress, find the corresponding solution for Sverdrup flow. Define dimensional scales and state when Eq. 2 approximately satisfies Sverdrup balance.

iv) Consider forcing by a point source, $F = \delta(\mathbf{x} - \mathbf{x}_0)$. For a uniform depth fluid under the β-plane approximation, sketch contours of ψ satisfying Eq. 2 far from boundaries but without assuming that r is small. Note, you do not need to obtain an explicit expression for ψ.

v) Consider a rectangular basin in the region $0 \leqslant x \leqslant L_x$, $0 \leqslant y \leqslant L_y$, where the fluid depth, $H(x)$, is a parabolic function of x and $H(x = 0) = H(x = L_x) = 0$. Invoking the β-plane approximation, find an expression for the pseudovelocity, $\widetilde{\mathbf{u}}$, defined above. Sketch contours of ψ associated with the steady state circulation associated with a sinusoidal wind stress curl $F = sin\,(2\pi y/L_y)$. Comment on the possible connection with your sketch and western boundary currents. You do not need to obtain an explicit expression for ψ.

END OF PAPER