UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 27 May, 2016 9:00 am to 11:00 am

PAPER 332

FLUID DYNAMICS OF THE SOLID EARTH

*You may attempt **ALL** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Magma is emplaced in a shallow, subsurface chamber with initial temperature T_∞ and bulk composition C_0. The magma is sub-eutectic, $C_0 < C_E$, and the composition of the solid is pure so that $C_s = 0$. Cooling through the country rock at the roof maintains a fixed temperature $T_B > T_E$ at the roof, where T_E is the eutectic temperature of the magma.

The enrichment of composition due to solidification of the fluid at the roof decreases the fluid density resulting in the formation of a stagnant mushy layer along the top boundary. Draw the temperature and composition of the liquid magma and mushy layer on a phase diagram using a linear liquidus relationship $T_L(C) = T_m - mC$.

Write down a complete set of equations governing the conservation of heat and composition in the liquid and mushy layer, along with appropriate boundary conditions to describe the evolution of the system. Derive, or state clearly, the terms in your equations for conservation of energy and composition within the mushy layer. You may neglect the diffusivity of composition within the mushy layer, but not within the liquid, and may take the heat capacity, thermal conductivity and density of the two phases as equal. Note that for most systems the compositional diffusivity is much smaller than the thermal diffusivity, $D \ll \kappa$. Find analytical solutions for the temperature, composition, porosity $\phi(z,t)$ and depth $h(t)$ of the mushy layer in the limit

$$\mathcal{C} = \frac{C_0}{\Delta C} \gg 1 \quad \text{with} \quad \mathcal{S} = \frac{L}{c_p \Delta T} = \mathcal{O}(\mathcal{C}),$$

where $\Delta C = C_B - C_0$, $\Delta T = T_0 - T_B$, $T_0 = T_L(C_0)$ and $T_L(C_B) = T_B$.

Finally, show that in the limit of negligible compositional diffusivity in the liquid, $D/\kappa \to 0$, the interface position is determined implicitly by the equation

$$\Omega e^{\lambda^2} \mathrm{erfc}\ \lambda = \theta_\infty e^{\Omega\lambda^2} \mathrm{erf}\ \Omega\lambda,$$

where $\lambda = h(t)/2\sqrt{\kappa t}$, and Ω should be written in terms of $\mathcal{S}$ and $\mathcal{C}$.

2

Buoyant CO_2 is injected along the horizontal and impermeable top surface of a two-dimensional groundwater aquifer of uniform permeability k and porosity ϕ, in which a large-scale pressure gradient drives a uniform background flow,

$$U = -\frac{k}{\mu}\frac{\partial p_0}{\partial x} = -\frac{k}{\mu}G,$$

where G the background pressure gradient, μ is the viscosity (you should assume equal viscosity of CO_2 and water), and x is the horizontal coordinate. The CO_2 spreads as a buoyant gravity current in the porous aquifer driven by both the density difference between CO_2 (ρ_c) and water (ρ_w) and by the background flow.

Use physical and mathematical arguments to construct a model of the evolution of the depth of the CO_2 gravity current $h(x, t)$ in the limit that the depth of the current is much less than the depth of the aquifer, $h \ll H$, and in which the extent of the current is very much greater than it's depth.

Consider first the case where CO_2 is injected at constant rate Q into the aquifer. Using a scaling argument, determine the early and late time behaviour of the porous current. At early times, find the scalings and determine a model equation describing the initially symmetric shape of the spreading CO_2 current. At late times, find the analytical expressions for the steady upstream profile and interior depth, and find the scaling and an approximate expression for the shape of the downstream nose.

Consider now the case where a constant volume V of CO_2 is injected rapidly and the volume then translates downstream while spreading due to gravity. Find the analytical, self-similar form of the constant volume CO_2 current.

3

The basal conditions of many glaciers are thought to play a crucial role in setting large scale patterns of flow. Consider the two-dimensional flow of a long, thin glacier, of Newtonian viscosity μ and density ρ, over a lubricating basal sediment. The upper surface of the glacier is stress free, and the precipitation of snow and surface melting are both negligible. At the horizontal base of the glacier the sediment exerts a shear stress on the ice of the form

$$\sigma = \mu \beta u_b,$$

where u_b is the basal ice velocity and β is a coefficient with units of length^{-1}.

Derive an expression for the velocity distribution within the ice that is driven by hydrostatic pressure gradients (within the ice) and limited by vertical shear stresses. By integrating over the depth, produce a model of the viscous flow of the ice, assuming that the total ice volume is conserved, and that the glacier had finite extent $x_N(t)$.

The initial ice thickness is approximately $10 \times 3/\beta$. Use a scaling analysis to determine the characteristic length, time and vertical scales of the current. Find analytical expressions for the early and late-time behaviour of the ice, calculating the approximate non-dimensional transition time between these two regimes.

Finally, find the profile of the steadily translating nose of the current of viscous, glacial ice as a function of the distance from the nose. How is the profile at the nose reflective of the dominant physical balances within the current at early and late times?

END OF PAPER