UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 26 May, 2016 1:30 pm to 4:30 pm

PAPER 331

HYDRODYNAMIC STABILITY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider Rayleigh-Bénard convection of an incompressible fluid with constant kinematic viscosity ν and thermal diffusivity κ between two stress-free horizontal boundaries, a vertical distance d apart, where the bottom boundary at $z = 0$ is held at temperature $T_0 + \Delta T$ and the top boundary is held at temperature T_0. The density ρ depends on temperature and obeys a linear equation of state, with $\rho = \rho_0$ when $T = T_0$.

(a) Calculate the conductive state for velocity, temperature and pressure. Using d, κ, ΔT and ρ_0 to non-dimensionalise, show that for small non-dimensional perturbations of (incompressible) velocity $\mathbf{u}$, pressure p and temperature θ away from the conductive state:

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} &= -\boldsymbol{\nabla} p + RaPr\theta\hat{\mathbf{z}} + Pr\nabla^2\mathbf{u}, \\ \frac{\partial \theta}{\partial t} - w &= \nabla^2\theta, \end{aligned}$$

where you should define the parameters Ra and Pr carefully.

(b) By considering the vorticity or otherwise, derive non-dimensional equations and stress-free boundary conditions for the perturbation vertical velocity w and the perturbation temperature θ. You may assume that:

$$w = W(z)X(x,y)e^{\sigma t}; \quad \theta = \Theta(z)X(x,y)e^{\sigma t}.$$

Show that

$$\begin{aligned} \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] X &= -\lambda^2 X, \\ \sigma\left(\frac{d^2}{dz^2} - \lambda^2\right) W &= -RaPr\lambda^2\Theta + Pr\left(\frac{d^2}{dz^2} - \lambda^2\right)^2 W, \\ \left(\frac{d^2}{dz^2} - \sigma - \lambda^2\right)\Theta &= -W, \end{aligned}$$

where λ is real. Hence derive an eigenvalue relation for σ. Prove that σ is real when $Ra > 0$ and $Re(\sigma) < 0$ when $Ra < 0$. Calculate the critical values of λ_c and Ra_c for marginal stability.

2

Consider infinitesimal two-dimensional perturbations about a parallel shear flow in an inviscid stratified fluid:

$$\begin{aligned}
\mathbf{u} &= \overline{U}(z)\hat{\mathbf{x}} + \mathbf{u}'(x,z,t), \\
p &= \overline{p}(z) + p'(x,z,t), \\
\rho &= \overline{\rho}(z) + \rho'(x,z,t), \\
\left[\mathbf{u}', p', \rho'\right] &= \left[\hat{\mathbf{u}}(z), \hat{p}(z), \hat{\rho}(z)\right]\exp[ik(x-ct)],
\end{aligned}$$

where the wavenumber k is assumed real, and the phase speed c may in general be complex.

(a) Applying the Boussinesq approximation appropriately, show that the vertical velocity eigenfunction $\hat{w}$ satisfies the Taylor-Goldstein equation:

$$\left(\frac{d^2}{dz^2} - k^2\right)\hat{w} - \frac{\hat{w}}{(\overline{U}-c)}\frac{d^2}{dz^2}\overline{U} + \frac{N^2\hat{w}}{(\overline{U}-c)^2} = 0; \; N^2 = -\frac{g}{\rho_0}\frac{d\overline{\rho}}{dz},$$

where N is the buoyancy frequency and ρ_0 is an appropriate reference density.

(b) Assume that there is a piecewise constant distribution of background density $\overline{\rho}$. Also assume that there is either a piecewise constant distribution or a piecewise linear distribution of background velocity $\overline{U}$. Show that the appropriate jump conditions at interfaces, where at least one of the density, vorticity or velocity are discontinuous, are given by:

$$\left[\frac{\hat{w}}{(\overline{U}-c)}\right]_-^+ = 0; \quad \left[(\overline{U}-c)\frac{d}{dz}\hat{w} - \hat{w}\frac{d}{dz}\overline{U} - \frac{g\overline{\rho}}{\rho_0}\left(\frac{\hat{w}}{(\overline{U}-c)}\right)\right]_-^+ = 0.$$

(c) Consider a three-layer flow:

$$\overline{U} = \begin{cases} \frac{\Delta U}{2} \\ \frac{\Delta U z}{h} \\ -\frac{\Delta U}{2} \end{cases}, \quad \overline{\rho} = \begin{cases} \rho_0 - \frac{\Delta\rho}{2} & z > \frac{h}{2}; \\ \rho_0 & |z| < \frac{h}{2}; \\ \rho_0 + \frac{\Delta\rho}{2} & z < -\frac{h}{2}. \end{cases}$$

You are given that $\tilde{c} = 2c/\Delta U$ satisfies

$$\tilde{c}^4 + \tilde{c}^2\left[\frac{e^{-4\alpha} - (2\alpha-1)^2}{4\alpha^2} - 1 - \frac{J}{\alpha}\right] + \left[\frac{(2\alpha - [1+J])^2 - e^{-4\alpha}(1+J)^2}{4\alpha^2}\right] = 0,$$

where $\alpha = kh/2$ and $J = g\Delta\rho h/[\rho_0 \Delta U^2]$. Hence show that the flow is unstable for

$$\frac{\alpha e^{\alpha}}{\cosh\alpha} < 1 + J < \frac{\alpha e^{\alpha}}{\sinh\alpha}.$$

Interpret this instability in terms of a wave resonance in the limit of large wavenumber, by considering the properties of the waves at each interface in isolation.

3

(a) Consider a general dispersion relation $D(k,\omega) = 0$ for wave-like perturbations proportional to $\exp[i(kx - \omega t)]$, where k and ω are in general complex. You may assume that the temporal growth rate has an unique maximum value $\omega_{i,\max}$ at some unique real wavenumber $k_{\max}$. For an observer travelling with velocity V along a ray $x/t = V$, you may also assume that the temporal growth rate $\sigma(V)$ perceived by this moving observer is

$$\sigma(V) = \omega_{\star,i} - V k_{\star,i},$$

where the in general complex wavenumber $k_\star$ is given by the saddle-point condition

$$\frac{\partial \omega}{\partial k}(k_\star) = \frac{x}{t},$$

and the in general complex frequency $\omega_\star$ is given by $D(k_\star, \omega_\star) = 0$.

(i) Define linear stability and linear instability in terms of $\omega_{i,\max}$.

(ii) Define the absolute wavenumber, absolute frequency and absolute growth rate.

(iii) Give criteria for when the flow is convectively unstable and absolutely unstable.

(b) Consider the linear complex Ginzburg-Landau equation:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\psi - \mu\psi - (1 + ic_d)\frac{\partial^2}{\partial x^2}\psi = 0.$$

You may assume that ψ describes a wave-like perturbation, and so is proportional to $\exp[i(kx - \omega t)]$ where k and ω are in general complex.

(i) Describe the physical processes modelled by the parameters U, c_d and μ.

(ii) Derive the dispersion relation.

(iii) In the $\mu - U$ half-plane with $U > 0$, identify the regions of stability, convective instability and absolute instability.

(c) Consider (non-dimensional) inviscid plane Couette flow between two horizontal impermeable boundaries at $z = \pm 1$, with background velocity $U = z$ for $-1 \leqslant z \leqslant 1$. You may assume that the vertical velocity perturbation $w' = \hat{w}(z)\exp[i(\alpha x - \omega t)]$ satisfies

$$\left(z - \frac{\omega}{\alpha}\right)\left(\frac{d^2\hat{w}}{dz^2} - \alpha^2 \hat{w}\right) = 0.$$

You may ignore potential issues associated with critical layers.

(i) Derive a complete set of eigenfunctions consistent with the appropriate boundary conditions.

(ii) Briefly explain how these eigenfunctions are consistent with the fact that inviscid plane Couette flow is linearly stable.

4

(a) The linearised Navier-Stokes equations for an infinitesimal velocity perturbation $\mathbf{u}_p(x, y, z, t)$ to a time-evolving base flow $\mathbf{U}(x, y, t) = (U(x, y, t), V(x, y, t), 0)$ are:

$$\begin{aligned}\frac{\partial \mathbf{u}_p}{\partial t} + (\mathbf{U}(t)\cdot\boldsymbol{\nabla})\mathbf{u}_p &= -\boldsymbol{\nabla} p_p - (\mathbf{u}_p\cdot\boldsymbol{\nabla})\mathbf{U}(t) + Re^{-1}\nabla^2\mathbf{u}_p,\\ \boldsymbol{\nabla}\cdot\mathbf{u}_p &= 0.\end{aligned}$$

Show that the adjoint evolution equation (relative to the usual energy norm) over a finite time interval $[0, T]$ is

$$\begin{aligned}\frac{\partial \mathbf{u}_d}{\partial \tau} &= \boldsymbol{\Omega}(-\tau)\times\mathbf{u}_d - \nabla\times(\mathbf{U}(-\tau)\times\mathbf{u}_d) - \boldsymbol{\nabla} p_d + Re^{-1}\nabla^2\mathbf{u}_d,\\ \boldsymbol{\nabla}\cdot\mathbf{u}_d &= 0,\end{aligned}$$

where $\tau = -t$, $\boldsymbol{\Omega} = \boldsymbol{\nabla}\times\mathbf{U}$, $\mathbf{u}_d$ is the adjoint velocity variable and p_d is the equivalent 'pressure' adjoint variable enforcing incompressibility. The boundary conditions for both sets of equations may be taken to be periodic in x and z at some horizontal extents $\pm L_x$ and $\pm L_z$ respectively, with the velocities and pressure gradients going to zero as $|y| \to \infty$, and you may assume that initial and terminal conditions relating $\mathbf{u}_d$ and $\mathbf{u}_p$ are consistent.

(b) Consider the toy model:

$$\frac{d\mathbf{x}}{dt} = |\mathbf{x}|\mathcal{N}\mathbf{x} + \mathcal{L}\mathbf{x};\ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix};\ \mathcal{N} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix};\ \mathcal{L} = \begin{pmatrix} -\frac{1}{Re} & 1 \\ 0 & -\frac{1}{4Re} \end{pmatrix},$$

where $Re \gg 1$. Define the orientation $\rho(t) = x_2(t)/x_1(t)$, and the energy $E(t) = (x_1^2 + x_2^2)/2$.

(i) Show that the nonlinear term involving $\mathcal{N}$ does not affect the growth or decay of energy $E(t)$.

(ii) Show that energy growth begins at the orientation ρ^+, and ends at the orientation ρ^-, where

$$\rho^{\pm} = 2Re\left[1 \pm \sqrt{1 - 1/Re^2}\right].$$

(iii) Find the general solution $\mathbf{x}(t)$ to the linear problem with $\mathcal{N} = 0$.

(iv) Calculate the time T^* at which the gain $G(t) = E(t)/E(0)$ attains its maximum value $G_{\max}$ across all possible initial conditions.

(v) Calculate $G_{\max}$, and hence determine the leading order scalings $T^* \sim ARe^n$ and $G_{\max} \sim BRe^m$ for n, m integers, and A, B real constants to be determined.

(vi) Briefly comment on this scaling result.

END OF PAPER