UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 27 May, 2016 1:30 pm to 4:30 pm

PAPER 330

FLUID DYNAMICS OF THE ENVIRONMENT

*You may attempt **ALL** questions,*
*although full marks can be achieve by good answers to **THREE** questions.*

*There are **FOUR** questions in total.*

Completed answers are preferred to fragments. The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider an inviscid, two-dimensional, stably stratified flow in the $x - z$ plane with buoyancy frequency $N(z)$ and mean velocity profile $\mathbf{U} = (U(z), 0)$ across an extensive range of sinusoidal hills of height $z = h(x)$ where $h(x) = h_0 \sin k_T x$. Here, x is horizontal, z is vertically upwards and both h_0 and k_T are constants.

(a) State the 'WKB approximation' as it relates to steady internal waves in a stratified flow. By linearizing the equations of motion about the mean flow profile, show that under the WKB approximation steady waves obey

$$m^2 = \left(\frac{N^2}{k^2 U^2} - \frac{U''}{k^2 U} - 1 \right) k^2,$$

where $\mathbf{k} = (k, m)$ is the wavenumber vector. For the particular shear profile $U(z) = U_0 + Sz$ with $U_0 > 0$, determine the maximum height that can be achieved by the waves for both the case $S < 0$ and the case $S > 0$. State any assumptions made and whether they are satisfied.

(b) Without invoking the WKB approximation, determine the wave field for the mean velocity profile

$$U(z) = \begin{cases} U_2 & z > H, \\ U_1 & 0 \leqslant z \leqslant H, \end{cases} \qquad (\star)$$

for constant U_1, U_2 with $U_1 > 0$. What is the upper bound on U_1 for waves to be generated? Show that the amplitude of the wave in $z > H$ is

$$\eta_2 = 2 \frac{\cos\theta_1 \sin\theta_1}{\cos\theta_2 \sin(\theta_1 + \theta_2)} h_0,$$

where θ_1 and θ_2 are the angles between lines of constant phase and the vertical in the two regions. Over what range of U_2 is this valid? Give the amplitude of any other waves present in the system. [You may neglect any reflections from $z = h(x)$.]

(c) Consider now an isolated obstacle on an otherwise flat plain. For $t < 0$ there is no mean flow ($U(z) = 0$), while for $t \geqslant 0$ the mean flow is given by ($\star$). Determine the time $t = T$ at which the permanent (steady) waves first reach $z = H$. Sketch the causality envelopes for $t = T$ and $t = 2T$. Sketch also the phase lines for the permanent waves.

2

In a turbulent buoyant plume, the mass flux, $\pi\rho_o Q$, the momentum flux, $\pi\rho_o M$ and the buoyancy flux, $\pi\rho_o F$ are defined as

$$Q = wb^2\,, \quad M = w^2 b^2\,, \quad F = \frac{g\Delta\rho}{\rho_o} wb^2\,,$$

in terms of the horizontally averaged values within the plume for the vertical velocity w, the density deficit $\Delta\rho$ relative to the environment and b the effective radius of the plume. Note, ρ_o is a constant reference density.

(a) Show that

$$\frac{dQ}{dz} = 2\alpha M^{1/2}\,, \quad \frac{dM}{dz} = \frac{FQ}{M}\,, \quad \frac{dF}{dz} = Q\frac{g}{\rho_o}\frac{d\rho_a}{dz},$$

where the horizontal entrainment velocity is given in terms of the vertical velocity within the plume by the relation $u = \alpha w$ for constant α and ρ_a is the environmental density.

(b) Suppose the ambient stratification is described by the buoyancy frequency

$$N^2(z) = -\frac{g}{\rho_o}\frac{d\rho_a}{dz} = N_s^2\left(\frac{z}{z_s}\right)^{\beta},$$

where N_s and z_s are constants. Show that

$$\frac{dF}{dz} = -QN_s^2\left(\frac{z}{z_s}\right)^{\beta}.$$

Seek a similarity solution of the form $Q_s = Q_o z^q$, $M_s = M_o z^m$, $F_s = F_o z^f$, and find expressions for q, m and f. By considering how Q, M and F vary with height, deduce that these solutions apply when $-4 < \beta < -8/3$.

(c) Define the new variables $\hat{Q} = Q/Q_s$, $\hat{M} = M/M_s$ and $\hat{F} = F/F_s$ and the variable $\xi = \ln(z)$. Show that

$$\frac{d\hat{Q}}{d\xi} = -q(\hat{Q} - \hat{M}^{1/2})\,, \quad \frac{d\hat{M}}{d\xi} = -m(\hat{M} - \hat{F}\hat{Q}/\hat{M})\,, \quad \frac{d\hat{F}}{d\xi} = -f(\hat{F} - \hat{Q}).$$

(d) The similarity solution corresponds to $\hat{Q} = \hat{M} = \hat{F} = 1$. Explore the stability of perturbations to these solutions by using the expansion

$$(\hat{Q}, \hat{M}, \hat{F}) = (1, 1, 1) + \epsilon(Q', M', F'),$$

where $\epsilon \ll 1$, to show that

$$\frac{dQ'}{d\xi} = -qQ' + \frac{1}{2}qM'\,, \quad \frac{dM'}{d\xi} = mQ' - 2mM' + mF'\,, \quad \frac{dF'}{d\xi} = fQ' - fF'\,.$$

As $\beta \to -8/3$, the eigenvalues tend to the limiting values $0, -1, -10/3$. Calculate the limiting values of f, q and m in the case $\beta \to -8/3$ and interpret this result. [It may be helpful to conider the physical implication of the limiting behaviour of the eigenvalues.]

3

The depth averaged equations for a gravity current with down slope speed u, depth h and reduced gravity g' are

$$\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} = Eu,$$

$$\frac{\partial g'h}{\partial t} + \frac{\partial ug'h}{\partial x} = 0,$$

$$\frac{\partial hu}{\partial t} + \frac{\partial hu^2}{\partial x} + \frac{1}{2}\frac{\partial h^2 g'}{\partial x}\cos\theta = hg'\sin\theta - cu|u|,$$

where c is the constant coefficient of the bottom-drag, g' is the reduced gravity of the current, $E > 0$ is a constant entrainment coefficient and θ the angle of the slope to the horizontal.

(a) In the case that there is a constant buoyancy flux, B, supplied at $x = 0$, show that there is a solution to the equations in which u is a constant. You may assume that $h = h_0$ at $x = 0$ and that $\sin\theta > \frac{1}{2}E\cos\theta$. Find expressions for $h(x)$ and $g‘(x)$, as well as u.

(b) Suppose now that the current is particle laden with the buoyancy derived from the suspension of particles in the flow. Show that the decrease in particle load in the current as it advances down the slope is given by

$$\frac{\partial g'h}{\partial t} + \frac{\partial B}{\partial x} = -v_s g',$$

where v_s is the fall speed of the particles and there is no resuspension. In the limit $v_s << \beta B^{1/3}$, show that to leading order B varies with the distance $x > 0$ according to the approximate relation

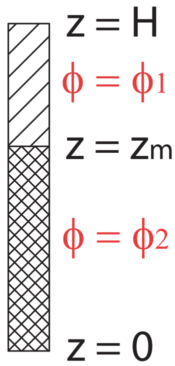
$$B \sim \left[B_o^{1/3} - \frac{v_s}{3\beta E}\ln\left(1 + \frac{Ex}{h_o}\right)\right]^3,$$

where $B = B_o$ and $h = h_o$ when $x = x_o$ and where

$$\beta = \left(\frac{\sin\theta - E\cos\theta}{C + E}\right)^{\frac{1}{3}}.$$

(c) Find the maximum distance travelled by the flow prior to sedimentation of all the particles.

(d) Describe the different circumstances under which B can increase with position down the slope in a particle laden current.



4

Consider a laminar suspension with a discontinuous particle concentration profile $\phi(z)$ in a container of height H. The identical particles have initial concentration ϕ_1 in $z_m < z \leqslant H$ and ϕ_2 in $0 < z \leqslant z_m$. There is no particle diffusion and the suspension is quiescent apart from particle settling and slow mixing. Particle settling is governed by $u(\phi) = u_s(1-\phi)$, where u_s is the settling velocity for isolated particles. Consider the settling process at the bottom as an instantaneous transition to $\phi = \phi_{max} = 1$ in the deposit and assume the deposit has a perfect square arrangement.

(a) For the case $z_m = \frac{H}{2}$, calculate the expected final deposit thickness $\bar{h} = h_c$ and the final depth-averaged concentration $\bar{\phi}$ using depth-averaging techniques across the entire height.

(b) Derive the general shock condition. Assume $0 < \phi_1 < \frac{1}{4}$, $\phi_2 = 2\phi_1$ and $z_m = \frac{H}{2}$. Determine the characteristics of the flow and shock velocities separating the different regions and sketch these on a space-time phase diagram. Distinguish the interaction between the clear fluid and the upper layer (shock velocity V_1), between the upper and lower layer (shock velocity V_2), and between lower layer and the settled particles in the deposit (shock velocity V_3). Calculate the time t_{c1} and height h_{c1} at which two shocks (which ones?) merge, creating a secondary shock with velocity V_4. Show that the process comes to rest after $t_{c2} = \frac{H(1-\phi_1-\phi_1 z_m/H)}{u_s(1-\phi_1)}$ when the particles reached a height $h_{c2} = \phi_1(H + z_m)$. Indicate the times t_{c1}, t_{c2} and heights h_{c1}, h_{c2} in your sketch.

(c) Now consider z_m as a free parameter. Determine the value of z_m for which the shocks with velocities V_1, V_2, V_3 form a triple point where all three shocks arrive at the same point in space and time. Calculate the time t_c and height h_c of this triple point and sketch this space-time diagram with time t_c, height h_c and the shocks drawn in. What happens at $t > t_c$?

(d) Consider now the case with $0 < \phi_1 < \frac{1}{2}$, $\phi_2 = \frac{1}{2}\phi_1$ and $z_m = \frac{4H}{5}$. Sketch the phase diagram, showing that the shocks with velocity V_1 (the interaction between the clear fluid and the upper layer) and velocity V_2 (the interaction between the lower layer and the final deposit) are constant up to h_{c1}, t_{c1} and h_{c2}, t_{c2}, respectively. Determine V_1, V_2, h_{c1}, t_{c1} and h_{c2}, t_{c2}. What happens at the interface between concentration ϕ_1 and ϕ_2? What can we say about the characteristic that joins point $z = z_m, t = 0$ and $z = h_c = \bar{h}, t = t_c$, with t_c the final deposition time? Determine time $t = t_c$. What other physical phenomenon might be associated with this concentration profile?

END OF PAPER