UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 31 May, 2016 9:00 am to 12:00 pm

PAPER 329

SLOW VISCOUS FLOW

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

State and prove the minimum dissipation theorem for Stokes flow, making it clear which flows are compared by the theorem.

Let $\mathbf{u}_0$ be a Stokes flow in a region V with velocity $\mathbf{V}_0(\mathbf{x})$ on ∂V and no body force, and let $\mathbf{u}$ be the flow produced by adding a rigid particle while maintaining $\mathbf{u} = \mathbf{V}_0(\mathbf{x})$ on ∂V. By using your proof of the minimum dissipation theorem, or otherwise, show that the increase $D' = D - D_0$ in dissipation due to the presence of the particle is given by

$$D' = \int_A (\mathbf{u}\cdot\boldsymbol{\sigma} - \mathbf{u}_0\cdot\boldsymbol{\sigma} - \mathbf{u}\cdot\boldsymbol{\sigma}_0)\cdot\mathbf{n}\,\mathrm{d}S,$$

where A is the surface of the particle, $\mathbf{n}$ is its inward normal, and $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_0$ are the stresses corresponding to $\mathbf{u}$ and $\mathbf{u}_0$. [Standard results may be quoted if helpful.] For the linear flow $\mathbf{u}_0 = \mathbf{U}_0 + \boldsymbol{\Omega}_0 \wedge \mathbf{x} + \mathbf{E}_0\cdot\mathbf{x}$ show further that

$$D' = \mathbf{F}\cdot(\mathbf{U} - \mathbf{U}_0) + \mathbf{G}\cdot(\boldsymbol{\Omega} - \boldsymbol{\Omega}_0) + \mathbf{S} : \mathbf{E}_0\,,$$

where $\mathbf{U}$ and $\boldsymbol{\Omega}$ are the velocity and angular velocity of the particle, and $\mathbf{F}$, $\mathbf{G}$ and

$$\mathbf{S} = -\tfrac{1}{2}\int_A (\mathbf{x}\,\boldsymbol{\sigma}\cdot\mathbf{n} + \boldsymbol{\sigma}\cdot\mathbf{n}\,\mathbf{x})\,\mathrm{d}S$$

are the force, couple and stresslet exerted by the particle.

Now let the particle be a thin straight rod of length $2L$, thickness ϵL and orientation $\mathbf{p}$ (with $|\mathbf{p}| = 1$), whose resistance to motion is given by the slender-body formula

$$\mathbf{f}(\mathbf{X}) = C(\mathbf{I} - \tfrac{1}{2}\mathbf{X}'\mathbf{X}')\cdot\big(\dot{\mathbf{X}} - \mathbf{u}_\infty(\mathbf{X})\big)\,,$$

where $C = 4\pi\mu/|\ln\epsilon|$ and $\mathbf{X}(s,t)$ is the position along the rod. Taking the centre of the rod to be $\mathbf{x} = \mathbf{0}$, calculate $\mathbf{F}$ and $\mathbf{G}$ for the linear flow $\mathbf{u}_0$ above.

Suppose $\mathbf{F} = \mathbf{G} = \mathbf{0}$. Show that $\mathbf{f} = -\tfrac{1}{2}Cs\,\mathbf{p}\cdot\mathbf{E}_0\cdot\mathbf{p}\,\mathbf{p}$ and interpret physically the dependence of this result on $\mathbf{p}$ and $\mathbf{E}_0$. Show also that $\mathrm{d}\mathbf{p}/\mathrm{d}t = \boldsymbol{\Omega}_0 \wedge \mathbf{p} + \mathbf{E}_0\cdot\mathbf{p} - \mathbf{p}\cdot\mathbf{E}_0\cdot\mathbf{p}\,\mathbf{p}$ and interpret this result physically. Calculate $\mathbf{S}$ for this case.

Now consider a force-free couple-free rod in a uniform shear flow. Let $\mathbf{u}_0 = (\gamma y, 0, 0)$ and $\mathbf{p} = \big(\cos\theta(t), \sin\theta(t), 0\big)$, where $\theta(0) = \pi/2$. Find and sketch the dependence of D' on θ, commenting on the physical interpretation of any maxima or zeroes. Find $\dot{\theta}$ and show that the total extra energy dissipated due to the particle in $-\infty < t < \infty$ has a finite value.

Speculate briefly on what you think would happen if there were a dilute suspension of such rods in a uniform shear flow.

2

The concentration C of surfactant on the surface of an inviscid bubble immersed in a very viscous fluid satisfies

$$\frac{DC}{Dt} = -C[\boldsymbol{\nabla}_s\cdot\mathbf{u}_s + (\mathbf{u}\cdot\mathbf{n})\boldsymbol{\nabla}_s\cdot\mathbf{n}] + D_s\nabla_s^2 C - k(C - C_0)\,, \tag{1}$$

where $\mathbf{n}$ is the unit normal out of the bubble; $\mathbf{u}_s = \mathbf{I}_s\cdot\mathbf{u}$, $\boldsymbol{\nabla}_s = \mathbf{I}_s\cdot\boldsymbol{\nabla}$ and $(\mathbf{I}_s)_{ij} = \delta_{ij} - n_i n_j$. Describe the physical interpretation of each of the terms in (1).

Assume that the steady concentration on a spherical bubble of radius a rising vertically with velocity $\mathbf{U}$ can be written as $C = C_0 + C'$, where $|C'| \ll C_0$ and C_0 is uniform. Derive an appropriately simplified form of (1) in the frame where the bubble is at rest. Explain why the velocity on the interface should be given by

$$\mathbf{u}(\mathbf{x}) = A\mathbf{I}_s(\mathbf{x})\cdot\mathbf{U}$$

for some constant A.

Show that $\boldsymbol{\nabla}_s\mathbf{n} = \mathbf{I}_s/a$ and derive expressions for $\boldsymbol{\nabla}_s\cdot\mathbf{n}$, $\nabla_s^2\mathbf{n}$ and $\boldsymbol{\nabla}_s\cdot\mathbf{I}_s\cdot\mathbf{U}$. Hence verify that

$$C' = B\mathbf{U}\cdot\mathbf{n}$$

and determine the constant B as a multiple of A. State conditions under which the assumption $|C'| \ll C_0$ is valid and give a physical interpretation of their meaning.

For $|C'| \ll C_0$ the surface-tension coefficient is given by $\gamma(C) = \gamma_0 - \gamma_1 C'$, where $\gamma_0 = \gamma(C_0)$ and γ_1 is a positive constant. Write down the general stress boundary condition for a fluid–fluid interface with surface tension γ and curvature κ, and show that in this case

$$\big[\mathbf{I}_s\cdot\boldsymbol{\sigma}\cdot\mathbf{n}\big]_-^+ = 6\mu A\lambda\,\mathbf{I}_s\cdot\mathbf{U}/a\ ,$$

where λ is a constant that should be identified.

Assuming that $\mathbf{u} \to -\mathbf{U}$ as $r/a \to \infty$, explain why the Papkovich–Neuber potentials for the flow can be written in the form

$$\boldsymbol{\Phi} = \mathbf{U} + \alpha a\,\mathbf{U}\,\frac{1}{r}\ , \qquad \chi = \beta a^3\,\mathbf{U}\cdot\boldsymbol{\nabla}\,\frac{1}{r}\,,$$

where α and β are constants. These potentials correspond to

$$\mathbf{u} = -(1 + \alpha + \beta)\mathbf{U} + (3\beta - \alpha)(\mathbf{U}\cdot\mathbf{n})\mathbf{n}\,, \tag{2}$$

$$\boldsymbol{\sigma}\cdot\mathbf{n} = \frac{12\mu}{a}\{\,\beta\mathbf{U} + (\alpha - 3\beta)(\mathbf{U}\cdot\mathbf{n})\mathbf{n}\,\} \tag{3}$$

on $r = a$. Use (2) and (3) to determine α, β and A in terms of λ.

Interpret the limits $\lambda \to 0$ and $\lambda \to \infty$ in terms of the surfactant effects on the tangential velocity and stress on the interface.

What balances the value of $\mathbf{n}\cdot\boldsymbol{\sigma}\cdot\mathbf{n}$ given by (3)?

3

A rigid horizontal plane $z = 0$ is covered with a thin layer of viscous fluid of uniform initial thickness h_0. A squeegee (window-wiper blade) is modelled as a thin vertical planar barrier of horizontal extent $2L$ in the y-direction and large vertical extent, whose lower edge is distance ϵh_0 above the plane, where $0 < \epsilon < 1$. The squeegee is moved horizontally with a constant velocity U perpendicular to its own plane. Surface tension is negligible.

(a) Use the equations of lubrication theory to derive the dimensionless evolution equation for the flow under gravity around the squeegee

$$\frac{\partial h}{\partial t} - \frac{\partial h}{\partial x} = \frac{1}{3}\boldsymbol{\nabla}\boldsymbol{\cdot}(h^3\boldsymbol{\nabla}h) \ ,$$

where $h \to 1$ as $x \to \infty$ and the squeegee is at $x = 0$, $-\alpha \leqslant y \leqslant \alpha$. The dimensionless variables should be defined. Assuming that the scales used to make the equations dimensionless are representative of the flow, express the conditions for lubrication theory to be appropriate in terms of the relevant parameters.

(b) Consider the limit $\alpha = \infty$ of an infinitely long squeegee. Find the steady-state fluid thickness in $x > 0$ in the implicit form $X(h) = x_0 - x$, where x_0 is a constant. What physical condition determines the value of x_0? Sketch the profile $h(x)$ in $x > 0$ when $x_0 \gg 1$.

The squeegee actually has a small dimensionless thickness 2δ, and the lower edge is a semi-circular arc of radius δ, where $\epsilon \ll \delta \ll 1$. Determine the flux under the squeegee in terms of the dimensionless pressure difference Δp across it. Hence show that the steady-state thickness just ahead of the squeegee is approximately

$$\frac{9\pi(2\delta)^{1/2}}{2\epsilon^{5/2}}$$

What is the thickness behind the squeegee?

(c) Consider the case $1 \ll \alpha < \infty$ of a long finite squeegee and assume that the flow is steady and ϵ is sufficiently small that the flux under the squeegee can be neglected. Sketch the expected form of the flow, showing the region where $h \gg 1$.

Assume that the steady-state thickness ahead of the squeegee has the approximate form $h(x, y) = \{9(x_N(y) - x)\}^{1/3}$ for $0 < x < x_N$, $|y| < \alpha$, where $1 \ll x_N \ll \alpha$. Why is this a reasonable assumption? By considering the steady flux balance in two dimensions, find an equation giving

$$\frac{\mathrm{d}}{\mathrm{d}y}\int_0^{x_N} h^4 \,\mathrm{d}x$$

as a function of y. Hence determine $x_N(y)$ and $h(0_+, y)$.

How small does ϵ have to be for the flux under the squeegee to be negligible?

4

Consider a viscous gravity current of density ρ_0 and viscosity $\lambda\mu$ spreading along the interface $z = 0$ between two viscous fluids of density ρ_1 in $z > 0$ and ρ_2 in $z < 0$. Both ambient fluids have viscosity μ, and $\rho_1 < \rho_0 < \rho_2$. The current is axisymmetric and occupies $h_2(r,t) < z < h_1(r,t)$, $r < R(t)$, and its thickness $h = h_1 - h_2$ satisfies $|\partial h/\partial r| \ll 1$. Let $H(t) = h(0,t)$ and $H(t) \ll R(t)$.

If the vertical force balance is hydrostatic, find the pressure in all three fluids and the relationship between h_1 and h_2. Why would you expect this relationship to hold, and what would happen if it didn't? Assuming that it does hold, and defining a modified pressure by subtracting off the hydrostatic pressure, show that the effect of gravity is equivalent to a radial body force $\mathbf{f}$ within the current of magnitude $\Delta\rho\, g\, \partial h/\partial r$, where $\Delta\rho$ is to be found.

Consider the flow driven by this body force. Use scaling arguments to show that, provided $H/R \ll \lambda \ll R/H$, the typical radial velocity in the current scales like $\Delta\rho\, gH^2/\mu$, explaining the two restrictions on λ. Use scaling arguments to show further that a current with fixed volume V spreads like $R \sim At^{1/5}$ and to determine the dependence of A on the other dimensional parameters. Deduce the form of the similarity solution for h.

The radial velocity in the current takes the form $At^{-4/5}U(\eta)$, where η is the similarity variable. Use the radial equation of mass conservation in the current to show that $U \propto \eta$.

If a thin rigid disc of radius R is placed at the origin of an axisymmetric straining flow $\mathbf{u}_\infty = E(r, 0, -2z)$ then it is known that the radial stress acting on each of its upper and lower surfaces is given by

$$\mathbf{e}_r \cdot \boldsymbol{\sigma} \cdot \mathbf{n} = \frac{8}{\pi} \frac{\mu E r}{(R^2 - r^2)^{1/2}} \ .$$

By considering the perturbation flow in this problem, deduce the value of $h\, \partial h/\partial r$ (or of the equivalent similarity function) in the self-similar gravity-current problem. What general properties of Stokes flow are you using to make this deduction?

Show that

$$R(t) = \left(\frac{125}{512\pi} \frac{\Delta\rho\, g V^2 t}{\mu} \right)^{1/5} \ .$$

Suppose now that the fluid in the gravity current differs from the upper fluid only by containing some dense dissolved chemical, with diffusivity D, that is insoluble in the lower fluid. Assume that the thickness of the current is now controlled by diffusion, i.e. $H \sim (Dt)^{1/2}$, and that the typical density difference $\Delta\rho(t)$ can be calculated from the initial density difference $\Delta\rho_0$ and volume V_0 of the current. Use scaling arguments to estimate $R(t)$ for this situation.

END OF PAPER