UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 6 June, 2016 9:00 am to 11:00 am

PAPER 328

BOUNDARY VALUE PROBLEMS FOR LINEAR PDES

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider the following initial-boundary value problem:

$$u_t = u_{xx} + \alpha u_x, \quad 0 < x < L, \quad 0 < t < T,$$

$$u(x,0) = u_0(x), \quad 0 < x < L,$$
$$u(0,t) = g_0(t), \quad u(L,t) = h(t), \quad 0 < t < T,$$

where $u_0(x)$, $g_0(t)$, $h(t)$ are given smooth functions and α, L, T, are given positive constants.

(i) Obtain an integral representation for the solution $u(x,t)$ in terms of $u_0(x)$, $g_0(t)$ and $h(t)$.

(ii) Discuss how the solution of this problem can be evaluated numerically for the case that the relevant integral transforms of u_0, g_0 and h can be computed explicitly.

2

The temperature $u(x,t)$ of a semi-infinite solid rod in contact at $x = 0$ with a tank filled with a certain liquid satisfies the following initial-boundary value problem:

$$u_t = u_{xx}, \quad 0 < x < \infty, \quad t > 0,$$

$$u(x,0) = u_0(x), \quad 0 < x < \infty,$$
$$u_t(0,t) + \alpha u_x(0,t) + \beta u(0,t) = f(t), \quad t > 0,$$

where α and β depend on the specific heat and conductivities of the rod and the liquid and $f(t)$ is the uniform temperature of the surrounding of the tank.

Obtain an integral representation of the solution in the particular case that

$$\beta = \frac{\alpha^2}{4}.$$

3

Let $u(x, y)$ denote a solution of the modified Helmholtz equation, namely of the equation

$$u_{xx} + u_{yy} - k^2 u = 0, \quad k > 0.$$

(i) Write a family of divergence forms involving a solution v of the adjoint equation.

(ii) Write a particular solution of v parameterised in terms of an arbitrary complex parameter λ.

(iii) Derive two global relations and show that if u is real, then the second global relation can be obtained from the first via complex conjugation followed by the transformation $\lambda \mapsto \bar{\lambda}$.

(iv) Consider the Dirichlet problem for the modified Helmholtz equation in the interior of the square with corners at the points

$$-1 + i, \quad -1 - i, \quad 1 - i, \quad 1 + i.$$

By expanding the boundary values in terms of appropriate basis functions obtain two approximate global relations.

(v) Determine four suitable sets of collocation points, each associated with each side. Show that these sets will yield a diagonaly dominant system of algebraic equations for the unknown coefficients in the expansions of (iv).

END OF PAPER