UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 27 May, 2016 1:30 pm to 3:30 pm

PAPER 327

DISTRIBUTION THEORY AND APPLICATIONS

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Define the spaces $\mathcal{D}(\mathbf{R}^n)$ and $\mathcal{D}'(\mathbf{R}^n)$, specifying the notion of convergence in each. Define the convolution between $\mathcal{D}'(\mathbf{R}^n)$ and $\mathcal{D}(\mathbf{R}^n)$.

Prove that $\mathcal{D}(\mathbf{R}^n)$ is dense in $\mathcal{D}'(\mathbf{R}^n)$.

Determine the limit in $\mathcal{D}'(\mathbf{R}^2 \setminus \{0\})$ of the sequence of functions

$$u_m(x_1, x_2) = m \sin\left(m|x_1^2 + x_2^2 - 1|\right), \quad m = 1, 2, 3, \ldots$$

Does the limit exist in $\mathcal{D}'(\mathbf{R}^2)$?

2

State and prove the Paley-Wiener-Schwartz theorem.

Let $\{y_m\}_{m=1}^N$ be a sequence of distinct points in $\mathbf{Z}^n$ and $\{f_m\}_{m=1}^N$ be a collection of entire, complex valued functions of $z \in \mathbf{C}^n$, none of which are identically zero. Suppose that the functions obey the estimates

$$\left|e^{\mathrm{i}z\cdot y_m} f_m(z)\right| \lesssim (1 + |z|)^m \exp\left(\tfrac{1}{m+1}|\mathsf{Im}\, z|\right), \quad m = 1, \ldots, N$$

for each $z \in \mathbf{C}^n$. Can you say anything with regards the linear independence of such a set of functions? Justify your answer.

3

Let $X \subset \mathbf{R}^n$ be open. Define the class of symbols $\mathrm{Sym}(X, \mathbf{R}^k; N)$. What does it mean for $\Phi : X \times \mathbf{R}^k \to \mathbf{R}$ to be a *phase function*?

If $a \in \mathrm{Sym}(X, \mathbf{R}^k; N)$ and Φ is a phase function explain how the oscillatory integral

$$I_\Phi(a) = \int e^{\mathrm{i}\Phi(x,\theta)} a(x, \theta)\, \mathrm{d}\theta,$$

defines a linear form on $\mathcal{D}(X)$. Show that $I_\Phi(a) \in \mathcal{D}'(X)$.

A distribution $u \in \mathcal{D}'(\mathbf{R}^2)$ is defined by

$$\langle u, \varphi \rangle = \int_{-\infty}^{\infty} x_2 \frac{\partial \varphi}{\partial x_1}(0, x_2)\, \mathrm{d}x_2.$$

Express u as an oscillatory integral. You should prove your answer does indeed give rise to the same distribution.

END OF PAPER