UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 7 June, 2016 1:30 pm to 3:30 pm

PAPER 326

INVERSE PROBLEMS

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet

Treasury Tag

Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Generalised inverses and regularisation of linear inverse problems
This question deals with the concepts of generalised inverses and regularisation.

(i) Recall the definition of the Moore-Penrose inverse.

(ii) Compute the Moore-Penrose inverse of the right-shift operator $K : \ell^2 \to \ell^2$, $\{u_j\}_{j\in\mathbb{N}} \to \{f_j\}_{j\in\mathbb{N}}$, with

$$f_j = (Ku)_j := \begin{cases} 0 & j = 1 \\ u_{j-1} & j \geqslant 2 \end{cases} .$$

It is necessary to also state the domain of $K^\dagger$.

(iii) Let $\mathcal{U}$ and $\mathcal{V}$ be Hilbert spaces. What is an equivalent condition to $f \in \mathcal{R}(K)$ for $K \in \mathcal{K}(\mathcal{U}, \mathcal{V})$?

(iv) Recall the definition of a regularisation (operator)? Give an example for a regularisation.

(v) We consider the problem of differentiation, formulated as the inverse problem of finding u from $Ku = f$ with the integral operator $K : L^2([0,1]) \to L^2([0,1])$ defined as

$$(Ku)(y) := \int_0^y u(x)\, dx .$$

Show that $R_\alpha : L^2([0,1]) \to L^2([0,1])$ with

$$(R_\alpha f)(x) := \frac{1}{\alpha} \begin{cases} f(x+\alpha) - f(x) & x \in \left[0, \frac{1-\alpha}{2}\right[\\ f(x+\frac{\alpha}{2}) - f(x-\frac{\alpha}{2}) & x \in \left[\frac{1-\alpha}{2}, \frac{1+\alpha}{2}\right[\\ f(x) - f(x-\alpha) & x \in \left[\frac{1+\alpha}{2}, 1\right] \end{cases}$$

for $\alpha \in]0, 1/2[$ is a convergent regularisation method and determine a corresponding a-priori parameter choice rule. In order to do so, verify the estimate

$$\|K^\dagger f - R_\alpha f^\delta\|_{L^2([0,1])} \leqslant \frac{\sqrt{6}}{\alpha}\delta + \frac{\sqrt{17}}{4}\alpha c$$

first, for $f \in H^2([0,1])$, $\|f''\|_{L^2([0,1])} \leqslant c$ and $f^\delta \in L^2([0,1])$ with $\|f - f^\delta\|_{L^2([0,1])} \leqslant \delta$. Without proof you are allowed to use the estimate

$$\int_0^{\frac{1-\alpha}{2}} |(R_\alpha(f))(x) - f'(x)|^2\, dx + \int_{\frac{1+\alpha}{2}}^1 |(R_\alpha(f))(x) - f'(x)|^2\, dx \leqslant \alpha^2 c^2 .$$

2 Bregman distances and error estimates

This question deals with error estimates of variational regularisation methods in the Bregman distance setting.

(i) Recall the definitions of the subdifferential for convex functionals and the Bregman as well as the symmetric Bregman distance.

(ii) Compute the Bregman distance $D_E(x, y)$ for E being the maximum-entropy regularisation

$$E(x) := \int_\Omega x(t)\log(x(t)) - x(t)\, dt\,,$$

for a bounded domain Ω. Use without proof that the subdifferential of a Fréchet-differentiable functional consists of its Fréchet-derivative only.

(iii) Draw a sketch of the Bregman distance $D_E^p(1, 0)$ for the function $E(x) := |x|$ and a $p \notin \{-1, 1\}$ of your choice.

(iv) Let $K \in \mathcal{L}(\mathcal{U}, \mathcal{V})$, and $J : \mathcal{U} \to \mathbb{R}$ be a convex, lower semi-continuous and proper functional. Further assume that there exist $u^\dagger \in \mathcal{U}$ and $f \in \mathcal{V}$ with $Ku^\dagger = f$. Show that the source condition

$$\mathcal{R}(K^*K) \cap \partial J(u^\dagger) \neq \emptyset\,, \tag{SC}$$

i.e. there exists an element $v \in \mathcal{U} \setminus \{0\}$ with $K^*Kv \in \partial J(u^\dagger)$, is equivalent to the existence of a function $\overline{u} \in \mathcal{U}$ such that $u^\dagger$ satisfies

$$u^\dagger \in \arg\min_{u\in\mathcal{U}} \left\{ \frac{1}{2} \|Ku - K\overline{u}\|_{\mathcal{V}}^2 + \alpha J(u) \right\}\,,$$

for $\alpha > 0$.

(v) Let the same assumptions hold true as in the previous exercise, but let $u^\dagger$ now be a J-minimising least-squares solution (for given data $f \in \mathcal{V}$). Verify the estimate

$$D_J^w(u_\alpha, u^\dagger) \leqslant D_J^w(u^\dagger - \alpha v, u^\dagger)\,,$$

for u_α being a solution of the Tikhonov-type regularisation functional

$$u_\alpha \in \arg\min_{u\in\mathcal{U}} \left\{ \frac{1}{2} \|Ku - f\|_{\mathcal{V}}^2 + \alpha J(u) \right\}\,,$$

and specify w.

(vi) How is the source condition (SC) connected to the generalised Eigenvalue problem? What are the consequences for the previously derived error estimate in case $u^\dagger$ is a generalised Eigenfunction of J and J is one-homogeneous? Does the result imply $u_\alpha = u^\dagger$?

3 Computational realisation of variational regularisation methods

This question deals with basic concepts in convex analysis and the computational realisation of convex, variational regularisation methods.

(i) Recall the definition of the proximity, respectively the resolvent operator, for a convex functional E.

(ii) Compute the Fréchet-derivative of

$$E(x) = \int_\Sigma y(t) \log \left(\frac{y(t)}{(Kx)(t)} \right) + (Kx)(t) - y(t) \, dt \, ,$$

for $K \in \mathcal{L}(\mathrm{PDF}(\Omega), L^1_+(\Sigma))$, bounded domains Ω and Σ, and $y(t) > 0$ for all $t \in \Sigma$. Without proof you are allowed to make use of the fact that the Fréchet-derivative E' satisfies

$$\frac{d}{d\tau} E(x + \tau z) \bigg|_{\tau = 0} = \langle z, E'(x) \rangle = \int_\Omega z(t) \, (E'(x))(t) \, dt \, ,$$

and that the order of differentiation (with respect to τ) and integration (with respect to t) can be interchanged.

(iii) Compute simple, closed-form solutions of the resolvent operators for the following convex functions or functionals:

- $E(x) = \frac{1}{2}\|DQx - y\|_2^2$, where $D \in \mathbb{R}^{m \times n}$ and $Q \in \mathbb{R}^{n \times n}$ are matrices such that $D^T D$ is a diagonal matrix, and Q is an orthogonal matrix.
- $E(x) = \int_\Omega y(t) \log \left(\frac{y(t)}{x(t)} \right) + x(t) - y(t) \, dt$, for a bounded domain Ω and $y(t) > 0$ for all $t \in \Omega$. Which one of the solutions makes sense and which one does not?
- $E(x) = |x|$.

(iv) The convex conjugate $E^* : \mathcal{X}^* \to \mathbb{R} \cup \{+\infty\}$ of a functional $E : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ is defined as

$$E^*(y) := \sup_{x \in \mathcal{X}} \langle x, y \rangle - E(x) \, .$$

Compute the convex conjugate E^* of the function $E(x) := \frac{\lambda}{2}|x - z|^2$, for a positive scalar $\lambda \in \mathbb{R}$.

(v) Show that the algorithm

$$\begin{aligned}
w^{k+1} &= \left(I + \frac{1}{\tau} \partial F^* \right)^{-1} \left(\frac{1}{\tau} u^k - D^T v^k \right) \\
u^{k+1} &= u^k - \tau \left(D^T v^k + w^{k+1} \right) \\
v^{k+1} &= (I + \sigma \partial G)^{-1} (v^k + \sigma D(u^{k+1} - \tau (D^T v^k + w^{k+1})))
\end{aligned}$$

is equivalent to the primal-dual hybrid gradient method (PDHGM) as introduced in the lecture, for a matrix $D \in \mathbb{R}^{m \times n}$ and convex functions $F^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$

and $G : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$. Without proof you are allowed to make use of the Moreau-identity

$$x = (I + \alpha\partial E)^{-1}(x) + \alpha \left(I + \frac{1}{\alpha}\partial E^* \right)^{-1} \left(\frac{x}{\alpha} \right) .$$

Conclude (from the lecture) how the parameters τ and σ have to be chosen in order to guarantee convergence.

END OF PAPER