UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 26 May, 2016 1:30 pm to 4:30 pm

PAPER 323

QUANTUM INFORMATION THEORY

Attempt no more than ***FOUR*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet

Treasury Tag

Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

In this question $\mathcal{E}^{\mathsf{B}\leftarrow\mathsf{A}}$ is an entanglement-breaking operation and $\mathcal{N}^{\mathsf{D}\leftarrow\mathsf{C}}$ is an arbitrary operation.

i. For systems A and B of finite Hilbert space dimension, any separable operator η_{AB} on $\mathcal{H}_{\mathsf{A}}\otimes\mathcal{H}_{\mathsf{B}}$ can be written $\eta_{\mathsf{AB}}=\sum_{i=0}^{k}\alpha(i)_{\mathsf{A}}\otimes\beta(i)_{\mathsf{B}}$ for some $k\in\mathbb{N}$ where the $\alpha(i)_{\mathsf{A}}$ and $\beta(i)_{\mathsf{B}}$ are positive semidefinite operators.

   Using this fact, show that

$$\mathcal{E}^{\mathsf{B}\leftarrow\mathsf{A}}=\mathcal{P}^{\mathsf{B}\leftarrow\tilde{\mathsf{Y}}}\mathcal{M}^{\tilde{\mathsf{Y}}\leftarrow\mathsf{A}} \tag{1}$$

   where $\mathcal{M}^{\tilde{\mathsf{Y}}\leftarrow\mathsf{A}}: L_{\mathsf{A}}\mapsto\sum_{y=0}^{k}|y\rangle\langle y|_{\tilde{\mathsf{Y}}}\mathrm{Tr}_{\mathsf{A}}E(y)_{\mathsf{A}}L_{\mathsf{A}}$ is the operation which measures a POVM E on A and stores the result in $\tilde{\mathsf{Y}}$, and $\mathcal{P}^{\mathsf{B}\leftarrow\tilde{\mathsf{Y}}}: L_{\tilde{\mathsf{Y}}}\mapsto\sum_{y=0}^{k}\tau(y)_{\mathsf{B}}\mathrm{Tr}_{\tilde{\mathsf{Y}}}|y\rangle\langle y|_{\tilde{\mathsf{Y}}}L_{\tilde{\mathsf{Y}}}$ is the operation which measures $\tilde{\mathsf{Y}}$ in the computational basis, obtaining result Y, and prepares B in a state $\tau(Y)_{\mathsf{B}}$.

ii. Let P_X be a distribution on a finite alphabet $\mathcal{A}_X$, let $\underline{\rho}$ be a map from $\mathcal{A}_X$ to states of AC, and let

$$\rho_{\tilde{\mathsf{X}}\mathsf{AC}}=\sum_{x}P_X(x)|x\rangle\langle x|_{\tilde{\mathsf{X}}}\otimes\underline{\rho}(x)_{\mathsf{AC}} \tag{2}$$

$$\sigma_{\tilde{\mathsf{X}}\tilde{\mathsf{Y}}\mathsf{D}}=\mathcal{M}^{\tilde{\mathsf{Y}}\leftarrow\mathsf{A}}\otimes\mathcal{N}^{\mathsf{D}\leftarrow\mathsf{C}}\rho_{\tilde{\mathsf{X}}\mathsf{AC}} \tag{3}$$

$$\omega_{\tilde{\mathsf{X}}\mathsf{BD}}=\mathcal{P}^{\mathsf{B}\leftarrow\tilde{\mathsf{Y}}}\sigma_{\tilde{\mathsf{X}}\tilde{\mathsf{Y}}\mathsf{D}}. \tag{4}$$

   Show that $I(\tilde{\mathsf{X}}:\mathsf{BD})_\omega=I(\tilde{\mathsf{X}}:\mathsf{B})_\omega+I(\tilde{\mathsf{X}}\mathsf{B}:\mathsf{D})_\omega-I(\mathsf{B}:\mathsf{D})_\omega$ and hence

$$\chi(\mathcal{E}^{\mathsf{A}\leftarrow\mathsf{B}}\otimes\mathcal{N}^{\mathsf{D}\leftarrow\mathsf{C}})\leqslant\chi(\mathcal{E}^{\mathsf{B}\leftarrow\mathsf{A}})+\chi(\mathcal{N}^{\mathsf{D}\leftarrow\mathsf{C}}),$$

   where χ is the Holevo information, stating clearly any results you use.

2

i. Let A and B be Hermitian operators on a Hilbert space of dimension d, and let I denote the identity operator on this space. Let $A = \sum_{0\leqslant j<d} \lambda_j |\alpha_j\rangle\langle\alpha_j|$ be an eigendecomposition where the λ_j are in non-increasing order.

Given $r \in \{0, \ldots, d\}$ show that, if $0 \leqslant B \leqslant I$ and $\mathrm{Tr}B = r$

$$\mathrm{Tr}AB \leqslant \sum_{0\leqslant j<r} \lambda_j. \tag{1}$$

ii. Let Q be a system of Hilbert space dimension d_Q, and let ρ_Q be a state of Q with eigendecomposition $\rho_\mathsf{Q} = \sum_{0\leqslant j<d_\mathsf{Q}} \lambda_j |\alpha_j\rangle\langle\alpha_j|$ where the λ_j are in non-increasing order. Suppose that there is a system K with Hilbert space dimension $k \leqslant d_\mathsf{Q}$ such that there exists an encoding operation $\mathcal{C}^{\mathsf{K}\leftarrow\mathsf{Q}}$ and a decoding operation $\mathcal{D}^{\mathsf{Q}\leftarrow\mathsf{K}}$ such that

$$F_{op}(\mathcal{D}^{\mathsf{Q}\leftarrow\mathsf{K}}\mathcal{C}^{\mathsf{K}\leftarrow\mathsf{Q}}, \rho_\mathsf{Q})^2 \geqslant 1 - \epsilon. \tag{2}$$

By considering Kraus representations of the encoding and decoding operation, show that $1 - \epsilon \leqslant \sum_{0\leqslant j<k} \lambda_j$.

3

i. Let L be a linear operator on Hilbert space $\mathcal{H}$ of finite dimension.

Define $|L|$ and define the trace norm $\|L\|_1$ and operator norm $\|L\|_{op}$ of L.

If L has polar decomposition $L = U|L|$ where U is a unitary operator on $\mathcal{H}$, show that if Z is a linear operator on $\mathcal{H}$ with $\|Z\|_{op} \leqslant 1$ then

$$\|L\|_1 \geqslant |\mathrm{Tr} ZL|.$$

ii. Let σ_{AB} be a state of AB where A and B are systems of Hilbert space dimension d, and let

$$\phi^+_{\mathsf{AB}} = \frac{1}{d} \sum_{0 \leqslant i,j < d} |i\rangle\langle j|_{\mathsf{A}} \otimes |i\rangle\langle j|_{\mathsf{B}}.$$

Show that if σ_{AB} has positive partial-transpose then $F(\phi^+_{\mathsf{AB}}, \sigma_{\mathsf{AB}}) \leqslant \frac{1}{\sqrt{d}}$.

iii. Suppose that

$$\rho_{\mathsf{AB}} = \frac{1}{2}\left(|0\rangle\langle 0|_{\mathsf{A}} \otimes |0\rangle\langle 0|_{\mathsf{B}} + |1\rangle\langle 1|_{\mathsf{A}} \otimes |1\rangle\langle 1|_{\mathsf{B}}\right) + \frac{\alpha}{2}\left(|0\rangle\langle 1|_{\mathsf{A}} \otimes |0\rangle\langle 1|_{\mathsf{B}} + |1\rangle\langle 0|_{\mathsf{A}} \otimes |1\rangle\langle 0|_{\mathsf{B}}\right).$$

For which values of $\alpha \in \mathbb{R}$ is ρ_{AB}

(a) a density operator?

(b) a pure state?

(c) a separable state?

4

i. State the Schmidt decomposition theorem.

ii. Suppose that

$$|\psi\rangle_{\mathsf{AB}} = \frac{1}{\sqrt{15}}\left(2\sqrt{2}|0\rangle_{\mathsf{A}} \otimes |0\rangle_{\mathsf{B}} - |0\rangle_{\mathsf{A}} \otimes |1\rangle_{\mathsf{B}} + \sqrt{2}|1\rangle_{\mathsf{A}} \otimes |0\rangle_{\mathsf{B}} + 2|1\rangle_{\mathsf{A}} \otimes |1\rangle_{\mathsf{B}}\right).$$

Write down $\psi_{\mathsf{A}} := \mathrm{Tr}_{\mathsf{B}}|\psi\rangle\langle\psi|_{\mathsf{AB}}$ and $\psi_{\mathsf{B}} := \mathrm{Tr}_{\mathsf{A}}|\psi\rangle\langle\psi|_{\mathsf{AB}}$ as matrices in the computational basis, and write down a Schmidt decomposition for $|\psi\rangle_{\mathsf{AB}}$.

iii. Let A and B be systems of Hilbert space dimension d, and let

$$|\phi^+\rangle_{\mathsf{AB}} = \frac{1}{\sqrt{d}} \sum_{0 \leqslant i < d} |i\rangle_{\mathsf{A}} \otimes |i\rangle_{\mathsf{B}}$$

Show that if $|\zeta\rangle_{\mathsf{AB}}$ is a state vector with Schmidt rank r then

$$F(|\phi^+\rangle\langle\phi^+|_{\mathsf{AB}}, |\zeta\rangle\langle\zeta|_{\mathsf{AB}}) \leqslant \sqrt{r/d}.$$

5

Let $\underline{\rho}$ be a map from a finite set $\mathcal{A}_X$ to the states of a system Q. Suppose that there is a subset $\mathcal{A}_M \subseteq \mathcal{A}_X$ of size k and a POVM $E : \mathcal{A}_M \to \mathcal{L}(\mathcal{H}_{\mathsf{Q}})$ such that

$$\frac{1}{k} \sum_{m \in \mathcal{A}_M} \mathrm{Tr} E(m)_{\mathsf{Q}} \underline{\rho}(m)_{\mathsf{Q}} = 1 - \epsilon.$$

Given a distribution P_X on $\mathcal{A}_X$ we define

$$\rho_{\tilde{\mathsf{X}}\mathsf{Q}} := \sum_{x \in \mathcal{A}_{\tilde{\mathsf{X}}}} P_X(x) |x\rangle\langle x|_{\tilde{\mathsf{X}}} \otimes \underline{\rho}(x)_{\mathsf{Q}}, \ \rho_{\tilde{\mathsf{X}}} := \mathrm{Tr}_{\mathsf{Q}} \rho_{\tilde{\mathsf{X}}\mathsf{Q}}, \text{ and } \rho_{\mathsf{Q}} := \mathrm{Tr}_{\tilde{\mathsf{X}}} \rho_{\tilde{\mathsf{X}}\mathsf{Q}}.$$

For $p \in [0,1]$, let $\omega[p]_{\tilde{\mathsf{Z}}} := (1-p)|0\rangle\langle 0|_{\tilde{\mathsf{Z}}} + p|1\rangle\langle 1|_{\tilde{\mathsf{Z}}}$.

1. Define the von Neumann entropy S and the quantum relative entropy D and give expressions for $S(\omega[p]_{\tilde{\mathsf{Z}}})$ and $D(\omega[p]_{\tilde{\mathsf{Z}}} \| \omega[q]_{\tilde{\mathsf{Z}}})$ in terms of p and q.

2. Show that there is a distribution P_X and a POVM $F : \{0,1\} \to \mathcal{L}(\mathcal{H}_{\tilde{\mathsf{X}}\mathsf{Q}})$ with result Z such that, if the state of $\tilde{\mathsf{X}}\mathsf{Q}$ is $\rho_{\tilde{\mathsf{X}}\mathsf{Q}}$ then $\Pr(Z=0) = 1-\epsilon$ and if the state of $\tilde{\mathsf{X}}\mathsf{Q}$ is $\rho_{\tilde{\mathsf{X}}} \otimes \rho_{\mathsf{Q}}$ then $\Pr(Z=0) = 1/k$.

3. Using the previous parts (or otherwise) show that

$$\log(k) \leqslant \frac{\sup_{P_X} I(\tilde{\mathsf{X}} : \mathsf{Q})_{\rho_{\tilde{\mathsf{X}}\mathsf{Q}}} + 1}{1-\epsilon}.$$

END OF PAPER