UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 2 June, 2016 9:00 am to 11:00 am

PAPER 322

BINARY STARS

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

A galaxy is 10 Gyr old and has been forming stars at a constant rate since its birth. Show that the fraction of stars younger than t is

$$Y(t) = 0.1\left(\frac{t}{\mathrm{Gyr}}\right).$$

Stars in the galaxy form with masses M chosen from an initial mass function

$$n(M)dM = \begin{cases} k\left(\frac{M_\odot}{M}\right)^3 dM, & M > 0.2\,M_\odot, \\ 0, & M < 0.2\,M_\odot, \end{cases}$$

where $n(M)dM$ is the number of stars with masses between M and $M + dM$ and k is a constant. Show that the fraction X of stars that form with mass greater than M is

$$X(M) = \frac{0.04}{(M/M_\odot)^2}, \quad M > 0.2\,M_\odot.$$

A star of mass M spends $8\,\mathrm{Gyr}/(M/M_\odot)^2$ on the main sequence and then a further $2\,\mathrm{Gyr}/(M/M_\odot)^2$ as a red giant before becoming a white dwarf. On a sketch of the (M, t) plane indicate the area that contains stars which are currently giants, giving formulae for its boundaries. Show that this maps to a triangle in the (X, Y) plane and hence that 0.5% of stars are currently giants and that 2% are currently white dwarfs.

All stars in the galaxy actually form in binary systems with the masses of the two components chosen independently from the above distribution. By considering the subdivisions of a cube, or otherwise, show that currently

(i) just under 1% of systems contain a red giant,

(ii) just over 3% of systems containing a red giant also contain another evolved star, either another red giant or a white dwarf and

(iii) for every binary system containing two red giants there are eight containing both a white dwarf and a red giant and sixteen containing two white dwarfs.

A second galaxy is similar to the first except that all stars formed in a single burst at a time $0 < t < 10\,\mathrm{Gyr}$ ago instead of continuously. What fraction of systems consist of two red giants now? For what range of starburst age t does this exceed the fraction of red giants seen now in the first galaxy?

2

In an Algol system a red giant of mass M_1 fills its Roche lobe of effective radius $R_{\rm L}$ and is transferring mass on to a main-sequence companion of mass M_2. The structure of the giant is such that its radius

$$R = f(L)M_1^{-0.27},$$

where $f(L)$ is a function of luminosity L only. Luminosity is itself a function only of conditions deep in the star and is independent of M_1. The effective Roche lobe radius is given by

$$\frac{R_{\rm L}}{a} = 0.462\left(\frac{M_1}{M}\right)^{1/3},$$

where $M = M_1 + M_2$ and a is the orbital separation. When the spin of the stars can be neglected show that the total angular momentum of the system is

$$J = \frac{M_1 M_2}{M} a^2 \Omega.$$

Thence show that conservative mass transfer ($\dot{M}_2 = -\dot{M}_1,\ \dot{J} = 0$) is dynamically stable only if the mass ratio

$$q = \frac{M_1}{M_2} < q_{\rm crit} \approx 0.7.$$

One way for the system to have reached its current state is for star 1 to have filled its Roche lobe before reaching its giant branch. Explain why this is so.

The radius of a star of mass M_1 at the base of its giant branch is given by

$$\frac{R_{\rm BGB}}{R_\odot} = 1.68\left(\frac{M_1}{M_\odot}\right)^{5/3}.$$

Show that, if star 1 is to have filled its Roche lobe before it became a giant then the initial orbital period $P_{\rm i}$ of the system must satisfy

$$P_{\rm i} < P_0\left(\frac{M_1}{M_\odot}\right)^2.$$

You may take $P_0 = 0.8$ days for the rest of this question.

If the mass transfer has been conservative show that the orbital period has evolved to maintain

$$P(M_1 M_2)^3 = \text{const.}$$

Hence show that P/M_1^2 would have had a minimum when $M_1 = \frac{5}{8}M$.

Algol now has an orbital period of 3 days and masses $M_1 = 1\,M_\odot$ and $M_2 = 3\,M_\odot$. Argue that it could well have formed via such conservative case A or early case B mass transfer.

Another semi-detached system with a mass-losing giant, RT Lac, has $P = 5$ days, $M_1 = 0.5\,M_\odot$ and $M_2 = 1.5\,M_\odot$. Show that it could not have formed conservatively.

Suggest how RT Lac might have evolved to its current state.

3

A binary system can be modelled as two point masses M_1 at $\mathbf{r}_1$ from the centre of momentum O and M_2 at $\mathbf{r}_2$ from O. Let $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, $r = |\mathbf{r}|$, $M = M_1 + M_2$ and $\mu = M_1 M_2 / M$. Show that the angular momentum of the system

$$\mathbf{J} = \mu \mathbf{h} = \mu \mathbf{r} \times \dot{\mathbf{r}},$$

energy

$$E = \frac{1}{2}\dot{\mathbf{r}}.\dot{\mathbf{r}} - \frac{GM\mu}{r}$$

and Laplace–Runge–Lenz vector $\mathbf{e}$ defined by

$$GM\mathbf{e} = \dot{\mathbf{r}} \times \mathbf{h} - \frac{GM\mathbf{r}}{r}$$

are conserved.

A single star of mass M moves through a cold medium of uniform density ρ in a straight line at constant speed v. Show that fluid initially at a distance d from the path of the star collides at a distance

$$b = \frac{d^2 v^2}{2GM}$$

behind the star and, assuming that the components of its motion perpendicular to the path of the star are eliminated in a shock, determine the remaining velocity of the fluid at this point.

Thence deduce that the accretion rate on to the star is

$$\dot{M} = \frac{4\pi (GM)^2 \rho}{v^3}.$$

In a circular binary system with orbital period P a star of mass M_1 loses mass at $-\dot{M}_1$ in a fast, isotropic, cold wind of speed v_{w}. Deduce the rate of accretion on to its companion of mass M_2.

END OF PAPER