UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 3 June, 2016 9:00 am to 11:00 am

PAPER 321

DYNAMICS OF ASTROPHYSICAL DISCS

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Viscous evolution and instability

(a) The evolution of a Keplerian accretion disk is governed by the diffusion equation

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}\left(r^{1/2}\overline{\nu}\Sigma\right)\right],$$

where Σ is the disk's surface density. The mean turbulent viscosity $\overline{\nu}$ depends on radius through $\overline{\nu} = \overline{\nu}_0\,(r/r_0)^{5/2}$, with $\overline{\nu}_0$ and r_0 constants. The viscous torque vanishes on the surface of the central object, taken to be at $r = 0$.

(i) Rewrite the diffusion equation in terms of dimensionless variables

$$x = \frac{r}{r_0}, \qquad \tau = 1 + \frac{3}{4}\frac{\overline{\nu}_0}{r_0^2}t,$$

and replace Σ by the dimensionless torque $g = \overline{\nu}\Sigma/(\overline{\nu}_0\Sigma_0)$, where Σ_0 is a constant reference density.

(ii) Suppose g depends solely on the similarity variable $\xi = \tau^{-1}x^{-1/2}$. Obtain g from the dimensionless diffusion equation, not forgetting to use the inner boundary condition. Subsequently, plot the surface density at time $\tau = 1$ and a later time.

(iii) Demonstrate that the total angular momentum is a constant but the total mass increases linearly with time. [*You may assume that the disk extends radially to infinity.*]

(b) In standard notation, the vertical structure of an accretion disc is described by the following equations

$$\frac{dp}{dz} = -\rho\Omega^2 z, \qquad \frac{dF}{dz} = \frac{9}{4}\mu\Omega^2,$$

$$\frac{dT}{dz} = -\frac{3\kappa\rho F}{16\sigma T^3}, \qquad p = \frac{k\rho T}{\mu_m m_p} + \frac{4\sigma T^4}{3c}.$$

(i) State what each of these equations represent. Assume that the gas pressure is negligible compared to the radiation pressure, and that the opacity is a constant. Show that the dynamical viscosity μ is also a constant.

(ii) In addition, suppose the alpha-prescription holds, so that $\mu \propto p$. By an order of magnitude treatment, derive the following scaling for the viscous torque: $\overline{\nu}\Sigma \sim \Sigma^{-1}$.

(iii) Finally, via a diagram (or otherwise) demonstrate that a localised overdensity grows and that the disk is hence viscously unstable.

2 Waves and 'planets' in the shearing sheet

The equations of a 3D compressible fluid in the shearing sheet are

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p - 2\Omega\mathbf{e}_z \times \mathbf{u} - \nabla\Phi_t, \qquad (1)$$

$$\partial_t \rho + \mathbf{u}\cdot\nabla\rho = -\rho\nabla\cdot\mathbf{u}, \qquad (2)$$

where $\mathbf{u}$, ρ, and p are the velocity, density, and pressure. The tidal potential is $\Phi_t = -(3/2)\Omega^2 x^2$, and the fluid is assumed barotropic, $p = p(\rho)$.

(a) Write down the linearised equations governing small axisymmetric 3D perturbations to the equilibrium: $\rho = \rho_0 =$ constant, $\mathbf{u} = -(3/2)\Omega\, x\, \mathbf{e}_y$.

(b) Assume that the perturbations are $\propto \exp(\mathrm{i}k_x x + \mathrm{i}k_z z - \mathrm{i}\omega t)$, where k_x and k_z are wavenumbers, and ω is a wave frequency. Hence derive the dispersion relation for waves in the shearing sheet:

$$\omega^4 - (\Omega^2 + c^2k^2)\omega^2 + \Omega^2 c^2 k_z^2 = 0,$$

where $k^2 = k_x^2 + k_z^2$, and $c^2 = dp/d\rho$ evaluated at $\rho = \rho_0$.

(c) By taking an appropriate limit, give an expression for the frequency of inertial waves. The phase velocity of a wave $\mathbf{c}_p$ is parallel to its wavevector, while the group velocity is defined by $\mathbf{c}_g = (\partial\omega/\partial k_i)\mathbf{e}_i$. Show that for inertial waves $\mathbf{c}_p \cdot \mathbf{c}_g = 0$.

(d) For the rest of the question suppose the fluid is polytropic, so that $p = K\rho^{1+1/n}$ where n is the polytopic index and K is a constant.

(i) Show that Equations (1) and (2) may be rewritten as

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla Q - 2\Omega\mathbf{e}_z \times \mathbf{u} - \nabla\Phi_t,$$

$$n(\partial_t Q + \mathbf{u}\cdot\nabla Q) = -Q\nabla\cdot\mathbf{u},$$

where $Q = (n+1)p/\rho$ is the enthalpy.

(ii) A steady model of a 2D 'planet' exhibits the following flow pattern:

$$\mathbf{u} = \alpha\, y\, \mathbf{e}_x - \beta\, x\, \mathbf{e}_y,$$

where α and β are positive constants. Compute the Q associated with this solution, and show that

$$\alpha = \frac{\beta^2 - 3\Omega^2}{\beta}.$$

(Note that Q is determined up to an arbitrary additive constant.) What restriction must be imposed on β for the solution to correspond to a discrete planet? Identify and give a mathematical expression for the shape of its physical boundary.

3 Maxwell's Adams Prize essay on Saturn's rings

Consider a large collection of identical particles located in a shearing sheet model of a self-gravitating planetary ring. The position of the nth particle is denoted by $[x_n(t),\, y_n(t)]$, and its motion is governed by

$$\begin{aligned} \ddot{x}_n - 2\Omega\dot{y}_n &= 3\Omega^2 x_n + f_x^n, \\ \ddot{y}_n + 2\Omega\dot{x}_n &= f_y^n, \end{aligned}$$

where an overdot indicates a time derivative and the gravitational acceleration arising from the surrounding particles is

$$\mathbf{f}^n = Gm \sum_{j\neq n} \frac{(x_j - x_n)\mathbf{e}_x + (y_j - y_n)\mathbf{e}_y}{[(x_j - x_n)^2 + (y_j - y_n)^2]^{3/2}}.$$

Here m is the mass of a particle and G is the gravitational constant.

(a) Show that an infinite row of equally spaced particles, described by

$$x_n = 0, \qquad y_n = hn, \qquad \dot{x}_n = \dot{y}_n = 0,$$

where h is a fixed length, is an equilibrium solution.

(b) Perturb this equilibrium by a small displacement $(x_n',\, y_n')$ and write down the linearised equations governing its evolution.

(c) Suppose the perturbation undergoes a collective Fourier motion so that

$$x_n' = X\exp(st + nhk\mathrm{i}), \qquad y_n' = Y\exp(st + nhk\mathrm{i}),$$

where X and Y are complex constants, s is a growth rate, and k is a wavenumber. Show that the formally infinite set of perturbation equations reduces to:

$$\begin{aligned} s^2X - 2\Omega sY &= 3\Omega^2X - \frac{Gm}{h^3}F(hk)\,X, \\ s^2Y + 2\Omega sX &= 2\frac{Gm}{h^3}F(hk)\,Y, \end{aligned}$$

where the function F is defined via

$$F(\xi) = 2\sum_{l=1}^{\infty} \frac{1-\cos(l\xi)}{l^3}.$$

The function $F(\xi)$ is 2π-periodic. Show that it achieves its maximum value when its argument ξ is an odd multiple of π.

(d) Write down and solve the dispersion relation for s. Prove that the ring is unstable if

$$\frac{Gm}{h^3\Omega^2} > \frac{13-4\sqrt{10}}{9F(\pi)}.$$

END OF PAPER