UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 1 June, 2016 1:30 pm to 4:30 pm

PAPER 320

GALACTIC ASTRONOMY AND DYNAMICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Suppose that any encounter of a galaxy with another galaxy leads to a merger if the impact parameter is less than R. Show that the probability of a merger in time T is

$$P = \pi R^2 \langle v_{\rm rel} \rangle NT,$$

where N is the number density of galaxies and $\langle v_{\rm rel} \rangle$ is the mean relative velocity.

By estimating R, N and $\langle v_{\rm rel} \rangle$ and recalling that 1 kpc $\approx 3 \times 10^{16}$ km, compute an order of magnitude estimate for the probability of a merger in a Hubble time.

Consider a perturbing mass $m_{\rm p}$ which passes a galaxy at impact parameter $\underline{p} = p\hat{\underline{x}}$ with a large velocity $\underline{v} = v\hat{\underline{z}}$. Here, $\hat{\underline{x}}$ and $\hat{\underline{z}}$ are unit vectors in the x-direction and the z-direction respectively. Suppose the centre of the galaxy is at the origin and the perturbing mass is in the (x, y) plane with position vector $\underline{r}_p$. For $r = |\underline{r}| < p$, show that the perturbing or tidal potential is

$$\psi(r) = \frac{Gm_{\rm p}}{{r_{\rm p}}^3}\left[-\frac{r^2}{2} + \frac{3(\underline{r}\cdot\underline{r}_{\rm p})^2}{2{r_{\rm p}}^2}\right].$$

Here, we are using the convention that the tidal force $= \nabla\psi$, whilst $r_{\rm p} = |\underline{r}_{\rm p}|$ is the distance of the perturber.

Hence, in the impulse approximation, show that a star at $\underline{r}$ receives a velocity increment

$$\Delta\underline{v} = \frac{2Gm_{\rm p}}{vp^2}\,(x, -y, 0)\,.$$

Assuming $\Delta\underline{v}$ is uncorrelated with $\underline{v}$, show that the change in energy per unit mass is

$$\Delta E = \frac{2G^2{m_{\rm p}}^2}{v^2p^4}(x^2 + y^2).$$

On averaging over a spherical galaxy of mass $m_{\rm g}$, deduce that

$$\Delta E = \frac{4G^2{m_{\rm p}}^2 m_{\rm g}}{3v^2p^4}\langle r^2 \rangle.$$

where $\langle r^2 \rangle$ is the mean square radius of stars in the galaxy.

Hence, deduce that in an encounter of two equal mass galaxies, the energy change is

$$\Delta E = \frac{8G^2{m_{\rm g}}^3}{3v^2p^4}\langle r^2 \rangle.$$

Explain why the orbital energy is

$$E = \frac{1}{4}m_{\rm g}v^2.$$

Show that the condition for a merger is

$$pv < \left[\frac{32}{3}G^2{m_{\rm g}}^2\langle r^2 \rangle\right]^{1/4}.$$

Explain why this formula fails when $v/p < \Omega$, where Ω is the circling frequency of stars in the galaxy.

2 Suppose a galaxy has density ρ and potential ψ (with the convention that the gravitational force $= \nabla\psi$). The spherical Jeans equations have the form:

$$\frac{d\rho\langle v_r^2\rangle}{dr} + \frac{\rho}{r}(2\langle v_r^2\rangle - \langle v_\theta^2\rangle - \langle v_\phi^2\rangle) = \rho\frac{d\psi}{dr},$$

where $\langle v_r^2\rangle, \langle v_\theta^2\rangle$ and $\langle v_\phi^2\rangle$ are the velocity second moments referred to the spherical polar coordinate system. Explain why $\langle v_\theta^2\rangle = \langle v_\phi^2\rangle$ for a spherical galaxy.

Show that the general solution of the Jeans equation for a spherical galaxy is

$$\rho\langle v_r^2\rangle = \int_0^\psi d\psi' \rho(r,\psi'),$$

$$\rho\langle v_\theta^2\rangle = \frac{1}{2r}\int_0^\psi d\psi' \frac{d}{dr}\left(r^2\rho(r,\psi')\right).$$

Let us consider a spherical galaxy of total mass M with potential

$$\psi(r) = \frac{GM}{(a^p + r^p)^{1/p}},$$

where a is a constant length-scale and p is a positive real number. Show that the density of the galaxy is

$$\rho(r) = \frac{(p+1)M}{4\pi}\frac{a^p}{r^{2-p}(a^p + r^p)^{2+1/p}}.$$

Using units such that $G = M = a = 1$, show that the density may be expressed as

$$\rho(r,\psi) = \frac{p+1}{4\pi} r^{p-2}\psi^{2p+1}.$$

Hence, deduce that the second velocity moments are

$$\langle v_r^2\rangle = \frac{1}{2(p+1)}\frac{1}{(1+r^p)^{1/p}},$$

$$\langle v_\theta^2\rangle = \frac{p}{4(p+1)}\frac{1}{(1+r^p)^{1/p}}.$$

By computing the kinetic energy T and the potential energy W of the model, show that the virial equation is satisfied at each location.

Let us denote the binding energy per unit mass as E and the absolute value of the angular momentum per unit mass as L. Demonstrate that the distribution function has the form

$$F(E,L) = CL^{p-2}E^{(3p+1)/2},$$

where C is a constant whose value is to be determined.

Is the distribution function unique ?

[*You are reminded of the standard integral* ($\mu > 0, \nu > 0$)

$$\int_0^{\pi/2} dy\, \sin^{2\mu-1} y \cos^{2\nu-1} y = \frac{\Gamma(\mu)\Gamma(\nu)}{2\Gamma(\mu+\nu)}. \quad]$$

3 Derive the formula for the relaxation time of a stellar system containing N stars as

$$T_{\rm relax} \approx 0.1 \frac{N}{\log N} T_{\rm cross},$$

where $T_{\rm cross}$ is the crossing time. Hence, show that the effects of stellar encounters are unimportant for most galaxies.

Define the phase space distribution function F and explain why it satisfies the collisionless Boltzmann equation.

A barred galaxy has a potential ψ steady in a frame rotating with pattern speed $\Omega_{\rm b}\,\underline{e}_z$, where $\underline{e}_z$ is a unit vector in the z-direction. Show that the equations of motion of a star confined to the (x, y) plane are

$$\begin{aligned} \ddot{R} - R\dot{\phi}^2 &= \frac{\partial \psi}{\partial R} + 2R\dot{\phi}\Omega_{\rm b} + \Omega_{\rm b}^2 R \\ R\ddot{\phi} + 2\dot{R}\dot{\phi} &= \frac{1}{R}\frac{\partial \psi}{\partial \phi} - 2\dot{R}\Omega_{\rm b} \end{aligned}$$

where (R, ϕ) are plane polar coordinates and superscripted dots represent time derivatives.

Show that the collisionless Boltzmann equation has the form

$$\frac{\partial F}{\partial t} + v_R \frac{\partial F}{\partial R} + \frac{v_\phi}{R}\frac{\partial F}{\partial \phi} + \left(2\Omega_{\rm b} v_\phi + \frac{v_\phi^2}{R} + \frac{\partial \psi_{\rm eff}}{\partial R}\right)\frac{\partial F}{\partial v_R}$$
$$+ \left(-2\Omega_{\rm b} v_R - \frac{v_R v_\phi}{R} + \frac{1}{R}\frac{\partial \psi_{\rm eff}}{\partial \phi}\right)\frac{\partial F}{\partial v_\phi} = 0$$

where (v_R, v_ϕ) are velocity components referred to polar coordinates in the rotating frame and $\psi_{\rm eff}$ is a suitable scalar potential that should be defined.

Show the zeroth moment is

$$\frac{\partial \rho}{\partial t} + \frac{1}{R}\frac{\partial (R\rho\langle v_R\rangle)}{\partial R} + \frac{1}{R}\frac{\partial \rho\langle v_\phi\rangle}{\partial \phi} = 0$$

where angled brackets denote averages over the distribution function.

Explain the physical meaning of the zeroth moment equation.

Now derive the Jeans equations for the system as:

$$\frac{\partial \rho\langle v_R\rangle}{\partial t} + \frac{\partial \rho\langle v_R^2\rangle}{\partial R} + \frac{1}{R}\frac{\partial \rho\langle v_R v_\phi\rangle}{\partial \phi} - 2\Omega_{\rm b}\rho\langle v_\phi\rangle + \frac{\rho(\langle v_R^2\rangle - \langle v_\phi^2\rangle)}{R} = \rho\frac{\partial \psi_{\rm eff}}{\partial R}$$
$$\frac{\partial \rho\langle v_\phi\rangle}{\partial t} + \frac{\partial \rho\langle v_R v_\phi\rangle}{\partial R} + \frac{1}{R}\frac{\partial \rho\langle v_\phi^2\rangle}{\partial \phi} + 2\Omega_{\rm b}\rho\langle v_R\rangle + \frac{2\rho\langle v_R v_\phi\rangle}{R} = \frac{\rho}{R}\frac{\partial \psi_{\rm eff}}{\partial \phi}$$

Give the physical meaning of the Jeans equations.

4 Examine the following statements. State whether they are true or false, giving mathematical arguments or physical reasoning to support your view.

(a) Evaporation is the process by which a series of weak encounters increases the energy of stars in a cluster until they finally escape. Therefore, a globular cluster whose evolution is dominated by evaporation becomes gradually less and less dense. This is because, as the stars slowly evaporate, the cluster is gradually losing mass.

(b) An isolated disc of stars is evolving through stellar dynamical processes alone. It is initially in a steady-state. Instabilities then act so as to restructure the distribution of stars in the disc. The end-point of the evolution of the disc is again a steady state, but now much more centrally concentrated. As the matter distribution has become more centrally concentrated, the magnitude of the potential energy $|W|$ of the system has increased. The kinetic energy T has also increased, because the total energy $E = T + W$ is conserved.

(c) The isothermal sphere has density ρ and relative potential ψ so that

$$\rho(r) = \frac{v_0^2}{4\pi G r^2}, \qquad \psi(r) = -v_0^2 \log r,$$

where v_0 is the circular speed. If all stars are moving on randomly oriented circular orbits, then the rms average speed is v_0. If the stars are all moving on eccentric orbits, then the rms average speed must also be v_0. This is because, for the same self-gravitating matter distribution, the total potential energy W is the same, and hence the total kinetic energy T must also be the same, via the virial theorem.

(d) Adiabatic invariance ensures that an initially closed periodic orbit will remain closed under slow changes of the potential. In the Keplerian potential, every bound particle follows an elliptical path and so is closed and periodic. Hence, if any change to the Keplerian potential is made sufficiently slowly, all orbits will remain closed and periodic, via the principle of adiabatic invariance.

(e) The circular speed of a star in a spherical galaxy or in the place of a razor-thin disc galaxy can be deduced from

$$\frac{v_{\rm circ}^2}{r} = \frac{GM(r)}{r^2} \implies v_{\rm circ} = \sqrt{\frac{GM(r)}{r}},$$

where $M(r)$ is the mass enclosed within radius r. Hence, a spherical galaxy and a razor-thin disc have the same rotation curve $v_{\rm circ}(r)$ if they have the same enclosed mass within radius r.

END OF PAPER