UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 30 May, 2016 9:00 am to 11:00 am

PAPER 318

MAGNETOHYDRODYNAMICS

*Attempt **ALL** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Conducting viscous fluid with density ρ, kinematic viscosity ν and magnetic diffusivity η flows steadily in the x-direction in the channel bounded by the planes $z = \pm L$ driven by a uniform pressure gradient $dp/dx = -\rho G$. There is a uniform magnetic field $B_0\hat{\mathbf{z}}$ applied across the channel. The boundary conditions at $z = \pm L$ are that the velocity vanishes and the magnetic field is normal to the boundary.

(i) Justify the assertion that the velocity $\mathbf{u}$ and induced magnetic field $\mathbf{b}$ (so that the total field is $B_0\hat{\mathbf{z}}+\mathbf{b}$) are of the form $\mathbf{u} = (u(z), 0, 0)$, $\mathbf{b} = (b(z), 0, 0)$, where u, b satisfy the coupled equations

$$0 = G + \frac{1}{\mu_0 \rho} B_0 \frac{db}{dz} + \nu \frac{d^2 u}{dz^2}, \quad 0 = B_0 \frac{du}{dz} + \eta \frac{d^2 b}{dz^2},$$

with $u(\pm L) = b(\pm L) = 0$.

(ii) Solve these equations for $u(z)$ and $b(z)$ and obtain the expressions

$$u = \frac{GL^2}{\nu H^2}(H \coth H)\left(1 - \frac{\cosh(Hz/L)}{\cosh H}\right),$$
$$b = \frac{B_0}{\eta}\frac{GL^3}{\nu H^2}\left(-\frac{z}{L} + \frac{\sinh(Hz/L)}{\sinh H}\right),$$

where H is the *Hartmann number* $H \equiv B_0 L/\sqrt{\mu_0 \rho \nu \eta}$. Derive an expression for the flux of fluid $Q = \int_{-L}^{L} u\, dz$ and sketch its form as a function of H keeping the other parameters fixed. Sketch also $u(z)$ and $b(z)$ when $H \gg 1$.

2

A simple model of Parker dynamo waves, with a fluctuating α-effect and with diffusion ignored, can be written in appropriate units as the pair of complex equations (with $D > 0$)

$$\frac{dA}{dt} = D^2 f(t) B, \qquad \frac{dB}{dt} = \mathrm{i}A,$$

where $f(t) = 1$, $0 < t < T$, $2T < t < 3T, \ldots$ and $f(t) = -1$, $T < t < 2T$, $3T < t < 4T, \ldots$

Consider the period $0 < t < T$. Defining $\sigma = D(1+\mathrm{i})/\sqrt{2}$, show that the general solution is

$$\begin{pmatrix} A \\ B \end{pmatrix} = P_+ \begin{pmatrix} \sigma \\ \mathrm{i} \end{pmatrix} \mathrm{e}^{\sigma t} + Q_+ \begin{pmatrix} -\sigma \\ \mathrm{i} \end{pmatrix} \mathrm{e}^{-\sigma t},$$

where P_+, Q_+ are complex constants. Deduce that

$$\begin{pmatrix} A(T) \\ B(T) \end{pmatrix} = \begin{pmatrix} \cosh \sigma T & -\mathrm{i}\sigma \sinh \sigma T \\ \frac{\mathrm{i}}{\sigma} \sinh \sigma T & \cosh \sigma T \end{pmatrix} \begin{pmatrix} A(0) \\ B(0) \end{pmatrix} \equiv \mathsf{M}^+ \begin{pmatrix} A(0) \\ B(0) \end{pmatrix}.$$

It may be assumed analogously that for the period $T < t < 2T$ we have the related results

$$\begin{pmatrix} A \\ B \end{pmatrix} = P_- \begin{pmatrix} \sigma^* \\ \mathrm{i} \end{pmatrix} \mathrm{e}^{\sigma^* t} + Q_- \begin{pmatrix} -\sigma^* \\ \mathrm{i} \end{pmatrix} \mathrm{e}^{-\sigma^* t},$$

where $\sigma* = -\mathrm{i}\sigma = D(1-\mathrm{i})/\sqrt{2}$ is the complex conjugate of σ, and

$$\begin{pmatrix} A(2T) \\ B(2T) \end{pmatrix} = \begin{pmatrix} \cosh \sigma^* T & -\mathrm{i}\sigma^* \sinh \sigma^* T \\ \frac{\mathrm{i}}{\sigma^*} \sinh \sigma^* T & \cosh \sigma^* T \end{pmatrix} \begin{pmatrix} A(T) \\ B(T) \end{pmatrix} \equiv \mathsf{M}^- \begin{pmatrix} A(T) \\ B(T) \end{pmatrix}.$$

Show that the matrices M^+, M^- have determinant 1.

Deduce that the mean exponential growth rate of A, B over very large times is given by $\ln\Lambda/2T$, where Λ is the larger eigenvalue of the matrix $\mathsf{N} = \mathsf{M}^-\mathsf{M}^+$. Determine the determinant and trace of N in terms of D, T and hence find Λ. Setting $T = 1$ show that as $D \to \infty$ the growth rate $\to (\sqrt{2}D - \ln 2)/2$.

3

Consider mean field dynamo action due to a time-dependent solenoidal velocity field

$$\boldsymbol{u} = \mathbb{R}\left(\boldsymbol{u}_1 \cos\Omega t + \boldsymbol{u}_2 \sin\Omega t\right)$$

where $\boldsymbol{u}_i = \hat{\boldsymbol{u}}_i \mathrm{e}^{\mathrm{i}\boldsymbol{k}_i\cdot\boldsymbol{x}}$ for distinct vectors $\boldsymbol{k}_1, \boldsymbol{k}_2$ with $|\boldsymbol{k}_1| = |\boldsymbol{k}_2| = k$. Assume that $\boldsymbol{B} = \boldsymbol{B_0} + \boldsymbol{b}(\boldsymbol{x}, t)$ where $\boldsymbol{B}_0$ is a constant vector, and that products of fluctuating quantities can be neglected in the equation for $\boldsymbol{b}$. Show that $\boldsymbol{b}$ takes the form

$$\boldsymbol{b} = \mathbb{R}\sum_{j=1,2} \mathrm{e}^{\mathrm{i}\boldsymbol{k}_j\cdot\boldsymbol{x}}(\boldsymbol{p}_j \cos\Omega t + \boldsymbol{q}_j \sin\Omega t)$$

and determine the $\boldsymbol{p}_j$ and $\boldsymbol{q}_j$.

Using the result (which should be justified)

$$\frac{1}{2}\mathbb{R}(\mathrm{i}\hat{\boldsymbol{u}}_j^* \times \hat{\boldsymbol{u}}_j) = H_j \boldsymbol{k}_j, \quad j = 1, 2$$

for some constants H_j, show that if the emf $\boldsymbol{\mathcal{E}} = \langle \boldsymbol{u} \times \boldsymbol{b} \rangle \equiv \boldsymbol{\alpha} \cdot \boldsymbol{B_0}$, (where $\langle\cdot\rangle$ denotes an average over space and time), then $\boldsymbol{\alpha}$ takes the form

$$\alpha_{ij} = \frac{\eta k^2}{2(\Omega^2 + \eta^2 k^4)}(H_1 k_{1i} k_{1j} + H_2 k_{2i} k_{2j}).$$

Give the form of $\boldsymbol{\alpha}$ when $\boldsymbol{k}_1 = (k, 0, 0)$, $\boldsymbol{k}_2 = (0, k, 0)$, $H_1 = H_2$. Is mean field dynamo dynamo action possible? Give brief reasons for your answer (detailed calculation is not required).

END OF PAPER