UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 27 May, 2016 1:30 pm to 4:30 pm

PAPER 316

PLANETARY SYSTEM DYNAMICS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

A planet on a circular orbit of semimajor axis $a_{\rm p}$ around a star of mass $M_\star$ is losing mass in the form of dust grains of diameter D. Sketch the ratio of the radiation pressure force to stellar gravity acting on the dust grains, β, as a function of D, identifying the different regimes on this plot and describing their physical origin.

Show that the new orbit of a dust grain with a given β has a semimajor axis $a_{\rm d}$ and eccentricity $e_{\rm d}$ given by

$$\begin{aligned} a_{\rm d}/a_{\rm p} &= (1-\beta)/(1-2\beta), \\ e_{\rm d} &= \beta/(1-\beta). \end{aligned}$$

If $\theta_{\rm d}$ is the azimuthal angle subtended at the star by the dust particle from its point of release, show that this increases at a rate $\dot{\theta}_{\rm d} = n_{\rm p}(1-\beta+\beta\cos\theta_{\rm d})^2$, where $n_{\rm p}$ is the mean motion of the planet.

If $\Delta\theta$ is the angular distance between the planet and the dust particle, for small β show that $\Delta\theta \approx 2\beta[\sin\theta_{\rm d} - \theta_{\rm d}]$.

Sketch $\Delta\theta$ as a function of $\theta_{\rm d}$ for 5 orbits of dust grains with $\beta = 0.01$ and 0.04.

Poynting-Robertson drag makes a dust particle migrate such that, for small β, its semimajor axes decays at a rate $\dot{a}_{\rm d} \approx -(2\beta G M_\star/c)a_{\rm d}^{-1}$, where c is the speed of light and G the gravitational constant. Show that by the time the particle has reached $\Delta\theta = 2\pi$, this drag force would have caused a fractional change in the particle's semimajor axis of $\Delta a_{\rm d}/a_{\rm d} \approx 2\pi v_{\rm p}/c$, where $v_{\rm p}$ is the orbital velocity of the planet.

Dust particles are continuously produced, but are quickly removed from the system such that in steady state the planet has a tail of particles with an azimuthal extent $\Delta\theta \ll 2\pi$. Suggest a physical mechanism which may be removing the particles.

If $n(\Delta\theta)d\Delta\theta$ is the number of particles in the tail in the range $\Delta\theta$ to $\Delta\theta + d\Delta\theta$, show that for small β and some other simplifying assumptions that should be stated, the tail has a density

$$n(\Delta\theta) \propto 1/\sin^2\left(\frac{\Delta\theta}{4\beta}\right).$$

2

Consider the motion of a test particle in the gravitational potential of a binary comprised of two bodies of mass M_1 and $M_2 \ll M_1$. These bodies follow a circular orbit about their centre of mass O; units are chosen such that both the distance between them and their mean motion are unity. The particle's motion is in the binary orbital plane and its location in this plane given by (x,y) in the rotating frame $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ that is centred on O with $\hat{\boldsymbol{x}}$ pointing towards M_2. Give expressions for r_1 and r_2, the distance of the particle from M_1 and M_2, respectively, in terms of x, y and $\mu_i = GM_i$, where G is the gravitational constant.

The equation of motion for the particle is

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \partial U/\partial x, \\ \ddot{y} + 2\dot{x} &= \partial U/\partial y, \end{aligned}$$

where $U = (x^2 + y^2)/2 + \mu_1/r_1 + \mu_2/r_2$. Show that

$$\begin{aligned} \partial U/\partial x &= \mu_1(1 - r_1^{-3})(x + \mu_2) + \mu_2(1 - r_2^{-3})(x - \mu_1), \\ \partial U/\partial y &= \mu_1(1 - r_1^{-3})y + \mu_2(1 - r_2^{-3})y. \end{aligned}$$

Sketch the centrifugal and gravitational accelerations experienced by the particle as a function of x for $y = 0$, and hence deduce that there are three collinear equilibrium points.

The collinear equilibrium point L_2 lies on the line connecting M_1 and M_2 on the other side of M_2 from M_1. Show that to lowest order in small quantities, the location of this point $(x_{L2},0)$ is such that its distance from M_2 is $r_{2,L2} \approx \alpha = \left(\frac{M_2}{3M_1}\right)^{1/3}$.

The particle is orbiting very close to L_2 at $y = Y$ and $x = x_{L2} + X$, where $X \ll \alpha$ and $Y \ll \alpha$. Show that the equation of motion can be written in the form $\dot{\mathbf{X}} = A\mathbf{X}$, where the vector $\mathbf{X} = [x, y, \dot{x}, \dot{y}]$ and the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1+2B & 0 & 0 & 2 \\ 0 & 1-B & -2 & 0 \end{pmatrix},$$

where B should be determined.

Hence show that L_2 is an unstable equilibrium point, and estimate the characteristic timescale for the distance of the particle from this point to grow.

3

Consider the evolution of a test particle due to the secular perturbations of an exterior planet of mass $M_{\rm p}$. Both objects orbit a star of mass $M_\star \gg M_{\rm p}$ in the same plane. The semimajor axis, eccentricity and longitude of pericentre of the particle are a, e, ϖ, and those of the planet $a_{\rm p}$, $e_{\rm p}$, $\varpi_{\rm p}$, respectively. Assuming all eccentricities to be small, the disturbing function is

$$\mathcal{R} \approx \left(\frac{GM_{\rm p}}{a_{\rm p}}\right)\left[\frac{1}{8}\alpha b^1_{3/2}(\alpha)e^2 - \frac{1}{4}\alpha b^2_{3/2}(\alpha)e_p e\cos(\varpi_{\rm p} - \varpi)\right],$$

where G is the gravitational constant, $\alpha = a/a_{\rm p}$ and $b^j_{3/2}(\alpha)$ are Laplace coefficients. Derive expressions for the evolution of the particle's eccentricity and longitude of pericentre using Lagrange's planetary equations $\dot{e} \approx -(\partial\mathcal{R}/\partial\varpi)/(na^2e)$ and $\dot{\varpi} \approx (\partial\mathcal{R}/\partial e)/(na^2e)$, where n is the particle's mean motion.

Hence show that the particle's complex eccentricity $z = e\exp(i\varpi)$ evolves as

$$\dot{z} = iA(\alpha)[b^1_{3/2}(\alpha)z - b^2_{3/2}(\alpha)z_{\rm p}],$$

where $A(\alpha) = (1/4)n_{\rm p}(M_{\rm p}/M_\star)\alpha^{1/2}$, $n_{\rm p}$ is the planet's mean motion, and $z_{\rm p}$ the planet's complex eccentricity.

The particle starts on a circular orbit and the planet's orbit is fixed. Show that its complex eccentricity after a time t is

$$z(t) = \left(\frac{b^2_{3/2}(\alpha)}{b^1_{3/2}(\alpha)}\right) z_{\rm p}[1 - \exp(iA(\alpha)b^1_{3/2}(\alpha)t)].$$

Describe this evolution on an Argand diagram.

Now consider the same problem, but in which the planet's mass is initially zero, growing linearly with time up to $M_{\rm p}$ over a timescale $t_{\rm p} = 2\pi/[A(\alpha)b^1_{3/2}(\alpha)]$, where $A(\alpha)$ is as defined above for the planet at its final mass. Derive an expression for the evolution of the particle's complex eccentricity for time up to $t_{\rm p}$, and comment on how the evolution has changed as a result of the changing planet mass.

Show that $z(t_{\rm p}) = 2z_{\rm p}[b^2_{3/2}(\alpha)/b^1_{3/2}(\alpha)]$.

Describe qualitatively the evolution of the particle's complex eccentricity had there been additional planets in the system, both with all planets having fixed masses, then again with one of the planets varying in mass.

4

Consider an asteroid belt of total mass $M_{\rm tot}$ made up of planetesimals with a dispersal threshold that is independent of their size D. Collision velocities are size independent and large enough that gravitational focussing can be ignored. The planetesimals are undergoing a collisional cascade with a size distribution that follows a power law with $n(D) \propto D^{-\alpha}$ between minimum and maximum sizes $D_{\rm min}$ and $D_{\rm max} \gg D_{\rm min}$, where $\alpha > 3$ is a constant and $n(D)dD$ is the number of planetesimals in the size range D to $D + dD$. Show that the collisional lifetime is $t_{\rm c} = AM_{\rm tot}^{-1}D^{\alpha-3}$, where A is a constant.

By considering the redistribution function to be scale independent, show that in steady state (which may be assumed) $\alpha = 7/2$.

The planetesimals are subject to the Yarkovsky force which removes them from the belt on a timescale given by $t_{\rm y} = BD + CD^{-1}$, where B and C are constants. Describe the physics of this force, and explain why this force is inconsequential for both small and large planetesimals.

Show that Yarkovsky timescales are less than collision timescales for some sizes as long as $M_{\rm tot} < M_{\rm y} = (3^{3/4}/4)AB^{-3/4}C^{-1/4}$, and on the same plot sketch the two timescales as a function of D for two different belt masses $M_{\rm tot} \gg M_{\rm y}$ and $M_{\rm tot} \ll M_{\rm y}$.

For the belt with $M_{\rm tot} \gg M_{\rm y}$, sketch the size distribution of material in the belt, and the size distribution in the population of objects removed by Yarkovsky forces. The slopes in the distributions, and the sizes at which any changes occur, should be quantified. You may assume that the removed material evolves due to dynamical processes that result in a lifetime that is independent of size.

Sketch and quantify these distributions for the belt with $M_{\rm tot} \ll M_{\rm y}$. You may assume that the redistribution function follows a power law with $\alpha = 7/2$.

END OF PAPER