UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 26 May, 2016 1:30 pm to 4:30 pm

PAPER 314

ASTROPHYSICAL FLUID DYNAMICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

You are reminded of the equations of ideal magnetohydrodynamics in the form

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \rho = -\rho \boldsymbol{\nabla} \cdot \boldsymbol{u} \,,$$

$$\frac{\partial p}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} p = -\gamma p \boldsymbol{\nabla} \cdot \boldsymbol{u} \,,$$

$$\rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} \right) = -\rho \boldsymbol{\nabla} \Phi - \boldsymbol{\nabla} p + \frac{1}{\mu_0} (\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B} \,,$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) \,,$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \,,$$

$$\nabla^2 \Phi = 4\pi G \rho \,.$$

1

Explain why the equations of ideal MHD, in the absence of gravity, and excluding the solenoidal constraint, can be written in the form

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}_i \frac{\partial \mathbf{U}}{\partial x_i} = \mathbf{0} \,,$$

where $\mathbf{U}$ is a multi-component vector and $\mathbf{A}_i$ is a square matrix for each Cartesian component i. [You need not write out the matrices $\mathbf{A}_i$ explicitly.]

Show that the wave speed v of any nonlinear simple wave in ideal MHD satisfies the inviscid Burgers equation, and use it to explain why such waves typically steepen and form shocks. Give a brief explanation of how shocks can be handled without explicitly including diffusive terms in the equations.

For a simple wave propagating in the x direction, calculate explicitly the eigenvectors of the matrix $\mathbf{A}_x$ corresponding to the Alfvén waves. Hence, or otherwise, show that there are special solutions of the MHD equations of the form

$$\boldsymbol{u} = u_x \, \boldsymbol{e}_x + f_1(x - vt) \, \boldsymbol{e}_y + f_2(x - vt) \, \boldsymbol{e}_z \,,$$
$$\boldsymbol{B} = B_x \, \boldsymbol{e}_x + f_3(x - vt) \, \boldsymbol{e}_y + f_4(x - vt) \, \boldsymbol{e}_z \,,$$

representing nonlinear Alfvén waves on a uniform background, where v, u_x, B_x, ρ and p are constant. Give expressions for v and the corresponding relations between the four functions f_i. Explain why these simple waves do not steepen.

2

Throughout this question, a non-self-gravitating perfect gas without a magnetic field undergoes isothermal flow, such that $p = c_{\rm s}^2\rho$ with $c_{\rm s} = \text{constant}$, by means of energy exchange with its surroundings.

Formulate the total energy equation for the gas, and show that, in order for it to remain isothermal, energy must be lost from the gas at a rate $-p\boldsymbol{\nabla}\cdot\boldsymbol{u}$ per unit volume (or to be supplied to the gas at a rate $+p\boldsymbol{\nabla}\cdot\boldsymbol{u}$).

Consider a planar *isothermal* shock in its rest frame. By considering the conservation of mass and momentum, show that the upstream density ρ_1, normal velocity u_1 and isothermal Mach number $\mathcal{M}_1 = u_1/c_{\rm s}$ are related to the downstream values ρ_2, u_2 and $\mathcal{M}_2$ by

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \mathcal{M}_1^2 = \frac{1}{\mathcal{M}_2^2}\,.$$

Show that, if isothermality is maintained by cooling to the surroundings (as opposed to heating by the surroundings), only compression shocks are physically acceptable.

Consider a steady, spherically symmetric, isothermal flow towards or away from a body of mass M. Show that a sonic point occurs at $r = r_{\rm s}$, where

$$r_{\rm s} = \frac{GM}{2c_{\rm s}^2}\,.$$

Show also that the isothermal Mach number $\mathcal{M} = |u_r|/c_{\rm s}$ and the scaled radial coordinate $x = r/r_{\rm s}$ are related by

$$\frac{1}{2}\mathcal{M}^2 - \ln\mathcal{M} = \frac{2}{x} + 2\ln x + \text{constant}\,.$$

Finally, show that the accretion rate of a steady transonic isothermal accretion flow is

$$\pi\,{\rm e}^{3/2}\frac{G^2M^2\rho_\infty}{c_{\rm s}^3}\,,$$

where ρ_∞ is the density of gas at rest at large distance.

3

Starting from Maxwell's equations, and explaining any assumptions made, derive the ideal induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B})$$

for the magnetic field in a conducting fluid.

Deduce carefully that (i) magnetic field lines can be identified with material curves, and (ii) the magnetic flux through an open material surface is independent of time. [If you use an equation for the time-evolution of a material surface element, you should derive it.]

In the process of star formation, a cloud of gas, threaded by a magnetic field, collapses through several orders of magnitude in size. Use a scaling argument to determine how the gravitational energy and the magnetic energy of the system depend on the characteristic length-scale L of the cloud, assuming that the mass M of the cloud and the magnetic flux Φ threading it are conserved during the collapse. Hence argue that, if the gas pressure is negligible, the cloud can collapse only if

$$\frac{M}{\Phi} > \frac{\lambda}{(\mu_0 G)^{1/2}},$$

where λ is a dimensionless number that need not be determined. Would you expect the pressure to become more or less important (relative to gravity and magnetic fields) as the collapse proceeds under (i) adiabatic or (ii) isothermal conditions?

4

A homogeneous incompressible fluid forms a uniform slab of thickness $2H$ and mass per unit area Σ under its own gravity.

Consider infinitesimal disturbances with (possibly complex) frequency ω and (positive) wavenumber k in the plane of the slab. Show that the displacement can be derived from a potential that satisfies Laplace's equation. Show also that the gravitational potential perturbation satisfies Laplace's equation both inside and outside the slab, but with a discontinuous gradient at the boundaries of the slab. By combining appropriate solutions of Laplace's equation in the interior and exterior of the slab and applying the relevant boundary conditions, show that disturbances with even symmetry satisfy the dispersion relation

$$\omega^2 = \frac{2\pi G\Sigma}{H}\left[kH\tanh(kH) - \frac{1}{1+\coth(kH)}\right].$$

What is the physical origin of the positive and negative contributions within the square brackets? Write down the corresponding relation for disturbances with odd symmetry.

By considering ω^2 as a function of $kH > 0$ in each case, discuss the stability of the equilibrium.

END OF PAPER