UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 6 June, 2016 9:00 am to 11:00 am

PAPER 313

APPLICATIONS OF DIFFERENTIAL GEOMETRY TO PHYSICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let g be a Euclidean metric on $\mathbb{R}^4$, and let vol be a volume form. Define the Hodge–$*$ operator of $(\mathbb{R}^4, g, vol)$ and find an expression for $*^2$ on $\Lambda^2(\mathbb{R}^4)$.

Define a decomposition of $\Lambda^2(\mathbb{R}^4)$ into self-dual (SD) and anti-self-dual (ASD) two–forms, and show that $H \wedge G = 0$ if H is SD and G is ASD. Deduce that the $SU(n)$ Yang–Mills action action on $\mathbb{R}^4$ is bounded from below by a multiple of the second Chern number.

Let $g = d\mathbf{x} \cdot d\mathbf{x} + dt^2$. Consider a gauge in which $A_t = 0$ vanishes to show that SD Yang–Mills equations take the form

$$\frac{\partial A_i}{\partial t} = -\frac{1}{2}\epsilon_{ijk} F_{jk}, \quad i, j, k = 1, 2, 3.$$

2

Let $A \in SL(n, \mathbb{R})$ and $M = \{B \in SL(n, \mathbb{R}), B^T = B\}$. What is the dimension of M? Show that $B \to ABA^T$ defines a group action of $SL(n, \mathbb{R})$ on M.

Assume that $n = 2$ and consider a basis

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

of $\mathfrak{sl}(2, \mathbb{R})$ to construct a representation of $\mathfrak{sl}(2, \mathbb{R})$ by vector fields $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ on a two-sheeted hyperboloid. Verify explicitly that $\mathbf{v}_i$ form $\mathfrak{sl}(2, \mathbb{R})$.

3

Write an essay on a theory of connections of principal bundles and its relationship to Yang–Mills instantons.

4

Define a Poisson structure on an n–dimensional manifold M in terms of an antisymmetric tensor field $\omega \in \Lambda^2(TM)$. Show that the Jacobi identity implies

$$\sum_{m=1}^{n} \omega^{mi}\frac{\partial \omega^{jk}}{\partial x^m} + \omega^{mk}\frac{\partial \omega^{ij}}{\partial x^m} + \omega^{mj}\frac{\partial \omega^{ki}}{\partial x^m} = 0, \tag{1}$$

where $(x^1, \ldots, x^n)$ is a local coordinate system on M, and $i, j, k, m = 1, \ldots, n$.

Let $\mathfrak{g}$ be an n–dimensional Lie algebra with a basis $(\mathbf{v}^1, \ldots, \mathbf{v}^n)$ and structure constants $c^{ij}{}_k$, i. e. $[\mathbf{v}^i, \mathbf{v}^j] = c^{ij}{}_k \mathbf{v}^k$. Define a Poisson bracket on M by

$$\{F, G\} = c^{ij}_k x^k \frac{\partial F}{\partial x^i}\frac{\partial G}{\partial x^j}, \quad \text{where} \quad F, G : M \to \mathbb{R}, \tag{2}$$

and show that equations (1) are satisifed.

Let $\mathfrak{g}$ be a Lie algebra generated by vector fields

$$\mathbf{v}^1 = y\partial/\partial z - z\partial/\partial y, \quad \mathbf{v}^2 = z\partial/\partial x - x\partial/\partial z, \quad \mathbf{v}^3 = x\partial/\partial y - y\partial/\partial x.$$

Write down the Hamilton equations corresponding to the Poisson structure (2) with the Hamiltonian $H = A(x^1)^2 + B(x^2)^2 + C(x^3)^2$, where A, B, C are constants.

END OF PAPER