UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 31 May, 2016 1:30 pm to 4:30 pm

PAPER 312

ADVANCED COSMOLOGY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Consider the 3+1 formalism for general relativity with the spacetime line element

$$ds^2 = -N^2dt^2 + {}^{(3)}g_{ij}(dx^i - N^i dt)(dx^j - N^j dt)\,, \qquad (*)$$

where $N(t,\, x^i)$ is the lapse function, $N^i(t,\, x^i)$ is the shift vector, and ${}^{(3)}g_{ij}(x^i)$ is the three-metric on constant time spacelike hypersurfaces Σ. (Latin indices $i = 1, 2, 3$.)

(i) The four-vector $n^\mu = \frac{1}{N}(1,\ N^i)$ is normal to Σ and defines the extrinsic curvature K_{ij} through its spacetime covariant derivative $K_{ij} \equiv -n_{i;j}$, for which you may assume the connection is given by $\Gamma^\mu_{\nu\lambda} = \frac{1}{2}g^{\mu\kappa}\left(g_{\nu\kappa,\lambda} + g_{\mu\kappa,\lambda} - g_{\nu\lambda,\kappa}\right)$.

Show that the extrinsic curvature can be expressed as

$$K_{ij} = -\frac{1}{2N}\left({}^{(3)}g_{ij,0} + N_{i|j} + N_{j|i}\right)\,,$$

where $|$ denotes the covariant derivative on Σ.

(ii) Consider the conformal 3-metric ${}^{(3)}\tilde{g}_{ij} = a^2(t,\, x^i)\,{}^{(3)}g_{ij}$ where $a^6 \equiv {}^{(3)}g = \det({}^{(3)}g_{ij})$ and, hence or otherwise, take the trace of the extrinsic curvature expression to find

$$K \equiv {}^{(3)}g^{ij}K_{ij} = -\frac{1}{2N}\left(\frac{{}^{(3)}\dot{g}}{{}^{(3)}g} + 2N^i_{|i}\right)\,.$$

In the context of an expanding universe (setting $N^i = 0$), argue that $-\frac{K}{3}$ can be interpreted as a locally defined Hubble parameter $H(t, x^i)$.

[*Hint: You may assume that* $\mathrm{Tr}(A^{-1}dA/dt) = d(\ln(\det A))/dt$ *for any matrix A with non-vanishing determinant.*]

(iii) In the long wavelength approximation in an expanding universe, we can approximate the traceless part of the extrinsic curvature with the general solution $\tilde{K}^i_j \approx C^i_j(x^k)\,a^{-3}$. Given this and results above, briefly motivate the choice of the following metric to describe nonlinear perturbations during inflation:

$$ds^2 = -[(1+\Psi)^2 - B_{,i}B^{,i}]dt^2 + 2a^2B_{,i}dt\,dx^i + a^2e^{2\zeta}\delta_{ij}dx^idx^j\,. \qquad (1)$$

(b) When linearising the 3+1 metric $(*)$ about a background (flat) FRW universe, we define the scalar perturbations by

$$N(t, x^i) \equiv \bar{N}(t)(1 + \Psi(t, x^i))\,, \qquad N_i \equiv -a^2B_{,i}\,, \qquad {}^{(3)}g_{ij} = a^2[(1-2\Phi)\delta_{ij} - 2E_{,ij}]\,,$$

$\rho = \bar{\rho} + \delta\rho$ and $P = \bar{P} + \delta P$, where bars denote background homogeneous quantities. Under the change of coordinates $(t,\ x^i) \longrightarrow (\tilde{t},\ \tilde{x}^i) = (t + \xi^0,\ x^i + \xi^i)$ (with $\xi^i \equiv \partial^i\lambda$), scalar metric perturbations transform as

$$\delta\tilde{g}_{ij} = \delta g_{ij} - \bar{g}_{ij,0}\xi^0 - \bar{g}_{kj}\xi^k_{,i} - \bar{g}_{ik}\xi^k_{,j}\,.$$

In synchronous gauge, we take $\Psi = 0$ and $B = 0$. Show that there is a residual gauge freedom in this gauge given by the coordinate transformation,

$$\xi^0 = \frac{C(x^i)}{\bar{N}}\,, \qquad \lambda = C(x^i)\int\frac{\bar{N}}{a^2}dt + D(x^i)\,,$$

where C and D are arbitrary functions of x^i only.

2

(i) For scalar perturbations about a spatially-flat Robertson–Walker metric in the conformal Newtonian gauge, the Boltzmann equation for the dimensionless temperature anisotropy of the CMB, Θ, can be written in Fourier space as the hierarchy

$$\dot{\Theta}_l + k\left(\frac{l+1}{2l+3}\Theta_{l+1} - \frac{l}{2l-1}\Theta_{l-1}\right) = -\dot{\tau}\left[(\delta_{l0}-1)\Theta_l - \delta_{l1}v_b + \frac{1}{10}\delta_{l2}\Theta_2\right] + \delta_{l0}\dot{\phi} + \delta_{l1}k\psi\,,$$

where v_b is the baryon velocity, ϕ and ψ are the metric potentials, and $\dot{\tau} = -a\bar{n}_e\sigma_{\mathrm{T}}$ is the differential optical depth (with $\bar{n}_e$ the unperturbed electron density, a the scale factor, and σ_{T} the Thomson cross section). Overdots denote differentiation with respect to conformal time η and k is the wavenumber. The $\Theta_l(\eta, \mathbf{k})$ are the angular moments of the Fourier transform of Θ at wavevector $\mathbf{k}$. Given that the energy density contrast of the CMB photons is $\delta_\gamma = 4\Theta_0$ and the bulk velocity $v_\gamma = -\Theta_1$, write down the continuity and Euler equations for the CMB.

If the baryons did not interact with the CMB, their Euler equation would be $\dot{v}_b + (\dot{a}/a)v_b + k\psi = 0$. By considering the total 3-momentum density of the photons and baryons, show that Thomson scattering modifies the baryon Euler equation to

$$\dot{v}_b + \frac{\dot{a}}{a}v_b + k\psi = \frac{\dot{\tau}}{R}(v_b - v_\gamma)\,,$$

where the quantity R should be specified.

(ii) Explain briefly what is meant by the tight-coupling approximation and show, to leading order in this approximation, that

$$\dot{v}_\gamma + \frac{\dot{R}}{1+R}v_\gamma + \frac{k}{4(1+R)}\delta_\gamma + k\psi = 0\,.$$

Hence show that the photon density contrast satisfies the oscillator equation

$$\ddot{\delta}_\gamma + \frac{\dot{R}}{1+R}\dot{\delta}_\gamma + \frac{k^2}{3(1+R)}\delta_\gamma = 4\ddot{\phi} + \frac{4\dot{R}}{1+R}\dot{\phi} - \frac{4}{3}k^2\psi\,. \qquad (*)$$

(iii) Define the angular power spectrum of the CMB temperature anisotropies, C_l.

Explain, with reference to approximate solutions of $(*)$, how the acoustic peaks arise in the angular power spectrum. Show that for adiabatic primordial perturbations, the peaks arise at multipoles

$$l \approx n\pi\chi_*/r_s(\eta_*)\,,$$

where n is a positive integer, χ_* is the comoving distance to the CMB last-scattering surface (which is at conformal time η_*), and $r_s(\eta_*)$ is the sound horizon at η_*. You should provide an expression for $r_s(\eta_*)$.

3

Consider tensor perturbations (gravitational waves) of a spatially-flat universe, for which the line element is

$$ds^2 = a^2(\eta)\left[-d\eta^2 + (\delta_{ij} + h_{ij})\, dx^i dx^j\right] , \qquad (*)$$

where $a(\eta)$ is the scale factor at conformal time η. The metric perturbation h_{ij} is symmetric and trace-free, and has vanishing divergence: $\partial_i h^i{}_j = 0$, where $h^i{}_j \equiv \delta^{ik} h_{kj}$. Throughout this question you should work to first order in the perturbation h_{ij}.

(i) A photon with energy ϵ/a and direction cosines $e^{\hat{i}}$, as measured relative to the orthonormal tetrad

$$(E_0)^\mu = a^{-1}\delta^\mu_0 , \quad (E_i)^\mu = a^{-1}\left(\delta^\mu_i - \frac{1}{2} h_i{}^j \delta^\mu_j\right) ,$$

has 4-momentum with coordinate components

$$p^\mu = \frac{\epsilon}{a^2}\left[1, e^{\hat{i}} - \frac{1}{2} h^i{}_j e^{\hat{j}}\right] .$$

Using the geodesic equation $dp^\mu/d\lambda + \Gamma^\mu_{\nu\rho} p^\nu p^\rho = 0$, with $p^\mu = dx^\mu/d\lambda$ and λ an affine parameter, show that the comoving energy of a photon, ϵ, satisfies

$$\frac{1}{\epsilon}\frac{d\epsilon}{d\eta} + \frac{1}{2}\dot{h}_{ij} e^{\hat{i}} e^{\hat{j}} = 0 ,$$

where overdots denote partial differentiation with respect to conformal time.

[*You may assume that the non-zero connection coefficients for the metric in* $(*)$ *are*

$$\begin{aligned}
\Gamma^0_{00} &= \mathcal{H} , \\
\Gamma^0_{ij} &= \mathcal{H}\delta_{ij} + \mathcal{H} h_{ij} + \frac{1}{2}\dot{h}_{ij} , \\
\Gamma^i_{j0} &= \mathcal{H}\delta^i_j + \dot{h}^i{}_j , \\
\Gamma^i_{jk} &= \partial_{(j} h^i{}_{k)} - \frac{1}{2}\delta^{il}\partial_l h_{jk} ,
\end{aligned}$$

where $\mathcal{H} = \dot{a}/a$ *is the conformal Hubble parameter and round brackets denote symmetrisation on the enclosed indices.*]

(ii) The Boltzmann equation for the dimensionless temperature anisotropy of the CMB, $\Theta(\eta, \mathbf{x}, \mathbf{e})$, at linear order in tensor perturbations is

$$\frac{\partial\Theta}{\partial\eta} + \mathbf{e}\cdot\boldsymbol{\nabla}\Theta - \frac{1}{\epsilon}\frac{d\epsilon}{d\eta} = \dot{\tau}\Theta - \frac{3\dot{\tau}}{16\pi}\int d\hat{\mathbf{m}}\,\Theta(\hat{\mathbf{m}})\left[1 + (\mathbf{e}\cdot\hat{\mathbf{m}})^2\right] ,$$

where $\dot{\tau}$ is the differential optical depth to Thomson scattering. Making the approximation $-\dot{\tau}e^{-\tau} = \delta(\eta - \eta_*)$, with the optical depth $\tau = 0$ at the present time η_0, show that the temperature anisotropy observed at $(\eta_0, \mathbf{x}_0)$ is

$$\Theta(\eta_0, \mathbf{x}_0, \mathbf{e}) \approx -\frac{1}{2}\int_{\eta_*}^{\eta_0} \dot{h}_{ij}(\eta', \mathbf{x}_0 - \chi\mathbf{e}) e^{\hat{i}} e^{\hat{j}}\, d\eta' , \qquad (**)$$

where $\chi = \eta_0 - \eta'$. Here, η_* is the time of last scattering. You should state clearly any further approximations that you make.

(iii) Consider a single gravitational plane wave of helicity $+2$ with wavevector $\mathbf{k} = k\hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ is a unit vector in the z-direction and k is the wavenumber. The corresponding metric perturbation is

$$h_{ij}(\eta, \mathbf{x}) \propto (\delta_{i1} + i\delta_{i2})\,(\delta_{j1} + i\delta_{j2})\, h^{(+2)}(\eta, k\hat{\mathbf{z}}) e^{ikz}\,,$$

where $h^{(+2)}(\eta, k\hat{\mathbf{z}})$ is the amplitude of the gravitational wave. The spherical multipoles of the observed temperature anisotropy at $\mathbf{x}_0 = \mathbf{0}$ are such that

$$\Theta(\eta_0, \mathbf{0}, \mathbf{e}) = \sum_{lm} \Theta_{lm} Y_{lm}(\mathbf{e})\,,$$

where $Y_{lm}(\mathbf{e})$ are the spherical harmonics. By expressing $\mathbf{e}$ in spherical polar coordinates, use $(**)$ to show that Θ_{lm} are only non-zero for $m = 2$.

For a very long wavelength gravitational wave, with $k\eta_0 \ll 1$, during matter domination up to the present time we may take

$$h^{(+2)}(\eta, k\hat{\mathbf{z}}) = h^{(+2)}(k\hat{\mathbf{z}}) \left[1 - \frac{1}{10}(k\eta)^2 + \cdots \right]\,,$$

where $h^{(+2)}(k\hat{\mathbf{z}})$ is the primordial amplitude of the wave. By expanding in $k\eta_0$, show that at leading order

$$\Theta_{22} \propto h^{(+2)}(k\hat{\mathbf{z}})(k\eta_0)^2 \quad \text{and} \quad \Theta_{32} = -\frac{i}{3\sqrt{7}}(k\eta_0)\Theta_{22}$$

for $\eta_0 \gg \eta_*$.

[*You may wish to use*

$$Y_{22}(\theta, \phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta e^{2i\phi}\,,$$
$$Y_{32}(\theta, \phi) = \frac{\sqrt{7}}{4}\sqrt{\frac{15}{2\pi}} \cos\theta \sin^2\theta e^{2i\phi}\,. \qquad]$$

4

(a) (i) Briefly explain the implications of Wick's Theorem for higher-order correlators of a Gaussian random field ζ_G.

(ii) A local non-Gaussian model is constructed by adding the square of a Gaussian random field ζ_G to itself as

$$\zeta(\mathbf{x}) = \zeta_{\rm G}(\mathbf{x}) + \tfrac{3}{5} f_{\rm NL}[\zeta_{\rm G}^2(\mathbf{x}) - \langle \zeta_{\rm G}^2(\mathbf{x}) \rangle] ,$$

where $f_{\rm NL}$ is the non-Gaussianity parameter. Define the bispectrum $B(k_1, k_2, k_3)$ and show, using the expression above, that the local-type bispectrum can be expressed as

$$B^{\rm loc}(k_1, k_2, k_3) = \tfrac{6}{5} f_{\rm NL} \big(P(k_1)P(k_2) + P(k_2)P(k_3) + P(k_3)P(k_1) \big) ,$$

given that the power spectrum $P(k)$ in Fourier space is defined by

$\langle \zeta_{\rm G}(\mathbf{k}_1) \zeta_{\rm G}(\mathbf{k}_2) \rangle = (2\pi)^3 P(k_1) \delta(\mathbf{k}_1 + \mathbf{k}_2)$ with $k = |\mathbf{k}|$.

(b) Using the in-in formalism during inflation, the leading order correction to an operator Q is given by the expectation value

$$\langle Q(t) \rangle = \mathcal{R}e \left\langle -2i Q^I(t) \int_{-\infty(1-i\mathcal{E})}^{t} H_{\rm int}^I(t')dt' \right\rangle , \qquad (\dagger)$$

where we will assume the interaction Hamiltonian $H_{\rm int}^I$ for single-field inflation at third-order is

$$H_{\rm int}^I = -M_{Pl}^2 \int d^3x a^3 \epsilon^2 \zeta \dot{\zeta}^2 , \qquad (\ddagger)$$

with slow-roll parameter ϵ (which you may assume is effectively constant) and scale factor given by $a \approx -1/(H\tau)$ with Hubble constant H and conformal time τ (i.e. $dt = a d\tau$) and $\dot{\zeta} = d\zeta/dt$ and $\zeta' = d\zeta/d\tau$. Assume that, in the interaction picture, the linear density perturbation ζ is a Gaussian random field with power spectrum,

$$\langle \zeta(\mathbf{k}, \tau) \zeta(\mathbf{k}', \tau) \rangle = (2\pi)^3 u_{\mathbf{k}}(\tau)\, u_{\mathbf{k}}^*(\tau)\, \delta(\mathbf{k} + \mathbf{k}') , \qquad (*)$$

where the mode functions $u_{\mathbf{k}}(\tau)$ and their conformal time derivatives are

$$u_{\mathbf{k}}(\tau) = \frac{H}{\sqrt{4\epsilon M_{\rm Pl}^2 k^3}} (1 + ik\tau)\, e^{-ik\tau} , \qquad u_{\mathbf{k}}'(\tau) = \frac{H}{\sqrt{4\epsilon M_{\rm Pl}^2 k^3}} k^2 \tau e^{-ik\tau} .$$

(i) Show that the three point correlator of ζ for the interaction Hamiltonian ($\ddagger$) reduces to the following terms,

$$\begin{aligned} \langle \zeta(\mathbf{k}_1, 0) \zeta(\mathbf{k}_2, 0) \zeta(\mathbf{k}_3, 0) \rangle = \mathcal{R}e \Big(& -2i \int d\tau \int d^3p_1\, d^3p_2\, d^3p_3 \\ \times\ & \frac{M_{Pl}^2 \epsilon^2}{(H\tau)^2} u_{\mathbf{k}_1}(0)\, u_{\mathbf{k}_2}(0)\, u_{\mathbf{k}_3}(0)\, u_{\mathbf{p}_1}(\tau)\, u_{\mathbf{p}_2}'(\tau)\, u_{\mathbf{p}_3}'(\tau) \\ \times\ & (2\pi)^3 \delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) \big[\delta(\mathbf{k}_1 + \mathbf{p}_1) \delta(\mathbf{k}_2 + \mathbf{p}_2) \delta(\mathbf{k}_3 + \mathbf{p}_3) + \text{cyclic perms.} \big] \Big) . \end{aligned}$$

(ii) Substitute the mode functions for the density field ζ and evaluate the integrals above explicitly to show that the three-point correlator becomes

$$\begin{aligned}\langle\zeta(\mathbf{k}_1,0)\zeta(\mathbf{k}_2,0)\zeta(\mathbf{k}_3,0)\rangle &= \\ &= (2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{H^4}{16\epsilon M_{\rm Pl}^4}\frac{1}{(k_1k_2k_3)^3}\left(\frac{k_2^2k_3^2}{K}+\frac{k_1k_2^2k_3^2}{K^2}+\text{cyclic perms.}\right),\end{aligned}$$

where $K = k_1 + k_2 + k_3$. Discuss any assumptions made in evaluating the integral.

(c) Consider the two bispectrum shapes given by the local-type non-Gaussian model $B^{\rm loc}(k_1, k_2, k_3)$ described in (a) and the single-field inflation bispectrum given in (b) which we shall denote as $B^{\rm sf}(k_1, k_2, k_3)$. By noting that the Planck satellite has obtained an observational limit on the local non-linearity parameter $|f_{\rm NL}| < 10$, discuss the prospects of obtaining a detection of the single-field bispectrum model (b).

[*Hint: Note that the local bispectrum signal is dominated by the squeezed limit (e.g.* $k_1 \ll k_2, k_3$ *with* $k_2 \approx k_3$*) and compare with the single-field bispectrum.*]

END OF PAPER