UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 3 June, 2016 1:30 pm to 4:30 pm

PAPER 311

BLACK HOLES

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a)(i) Explain how to define ingoing Eddington-Finkelstein coordinates (v, r, θ, ϕ) for the Schwarzschild solution and how to analytically extend the solution into a region $0 < r \leqslant 2M$. (ii) Determine the behaviour of radial null geodesics in these coordinates and hence plot a Finkelstein diagram. [10]

(b) Explain what is meant by "geodesic completeness" of a spacetime. On a Penrose diagram for the Kruskal spacetime, draw (i) an extendible radial null geodesic; (ii) an inextendible, complete, radial timelike geodesic; (iii) an inextendible, incomplete, radial null geodesic. [5]

(c) Alice is an astronaut who falls radially into a Schwarzschild black hole. During her fall, she sends radio signals radially to her friend Bob, who is at constant radius far from the black hole.

(i) Use outgoing Eddington-Finkelstein coordinates (u, r, θ, ϕ) to show that the redshift of the signals received by Bob is given by

$$\frac{\lambda_B}{\lambda_A} = \frac{du}{d\tau}$$

where Alice's trajectory is $u = u(\tau)$, $r = r(\tau)$, τ is her proper time, and the RHS is evaluated at the time the signal is emitted. [5]

(ii) As Alice approaches the event horizon, Bob observes that the redshift of the signals is proportional to e^{at} for some constant a, where t is Schwarzschild time. Determine the mass of the black hole in terms of a. [10]

2

(a) What is a null hypersurface? Prove that the normal to a null hypersurface is geodesic. [5]

(b) What is a trapped surface? State Penrose's theorem. Give an example of a spacetime containing trapped surfaces and explain briefly how it is consistent with Penrose's theorem. [6]

(c) Using a Penrose diagram, given an example of a spacetime constructed from initial data (Σ, h_{ab}, K_{ab}) for which (Σ, h_{ab}) is inextendible but the maximal Cauchy development is extendible. Explain why your example is consistent with the strong cosmic censorship conjecture. [4]

(d) At any point p one can choose normal coordinates (x^0, x^i) $(i = 1, 2, 3)$ such that $g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$.

(i) Prove that the dominant energy condition at p is equivalent to the condition that $T_{00} \geqslant \sqrt{T_{0i}T_{0i}}$ in any normal coordinate chart at p. [5]

(ii) A Maxwell field has energy momentum tensor

$$T_{ab} = \frac{1}{4\pi}\left(F_{ac}F_b{}^c - \frac{1}{4}F^{cd}F_{cd}\, g_{ab}\right)$$

In normal coordinates at p define $E_i = -F_{0i}$ and $B_i = (1/2)\epsilon_{ijk}F_{jk}$. Hence prove that a Maxwell field satisfies the dominant energy condition. [4]

(iii) Consider a scalar field with action

$$S = \int d^4x\sqrt{-g}F(X)$$

where $X = -(1/2)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. Prove that this scalar field satisfies the dominant energy condition if the function $F(X)$ satisfies the following conditions for all X:

$$F'(X) \geqslant 0 \qquad XF'(X) - F(X) \geqslant 0$$

[6]

3

(a) In a globally hyperbolic spacetime one can introduce coordinates (t, x^i) such that

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

The Einstein-Hilbert action is then (neglecting surface terms)

$$S = \int dt d^3x \sqrt{h} N \left({}^{(3)}R + K_{ij}K^{ij} - K^2 \right)$$

where ${}^{(3)}R$ is the Ricci scalar of h_{ij} and

$$K_{ij} = \frac{1}{2N}\left(\partial_t h_{ij} - D_i N_j - D_j N_i\right) \qquad K = h^{ij}K_{ij}$$

where D_i is the Levi-Civita connection of h_{ij}.

(i) Why are N and N^i non-dynamical fields? Determine the momentum π^{ij} conjugate to h_{ij}. [4]

(ii) Show that the Hamiltonian of General Relativity is (neglecting surface terms)

$$H = \int d^3x \sqrt{h}\left(N\mathcal{H} + N^i \mathcal{H}_i\right)$$

where $\mathcal{H}$ and $\mathcal{H}_i$ should be expressed in terms of h_{ij} and π^{ij}. [8]

(iii) Why does a closed universe have zero energy? [3]

(b) A Kerr black hole has metric

$$\begin{aligned} ds^2 &= -\frac{(\Delta - a^2\sin^2\theta)}{\Sigma}dt^2 - 2a\sin^2\theta\frac{(r^2+a^2-\Delta)}{\Sigma}dtd\phi \\ &+ \left(\frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \end{aligned}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta \qquad \Delta = r^2 - 2Mr + a^2$$

and $M > a > 0$.

(i) Let $D = g_{t\phi}^2 - g_{tt}g_{\phi\phi}$. Given that $D > 0$, show that $-g^{ab}(dt)_b$ is everywhere timelike and hence defines a time orientation. Deduce that t must increase along any future-directed causal curve. (Consider only the region outside the black hole and ignore the coordinate singularities at $\theta = 0, \pi$.) [5]

(ii) Consider an arbitrary causal curve. Prove that this must satisfy

$$\frac{-g_{t\phi} - \sqrt{D}}{g_{\phi\phi}} \leqslant \frac{d\phi}{dt} \leqslant \frac{-g_{t\phi} + \sqrt{D}}{g_{\phi\phi}}$$

[7]

(iii) Deduce that $d\phi/dt > 0$ for any causal curve in the ergosphere. [3]

4

(a)(i) Raychaudhuri's equation is

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \hat{\sigma}^{ab}\hat{\sigma}_{ab} + \hat{\omega}^{ab}\hat{\omega}_{ab} - R_{ab}U^aU^b$$

Show that if $\theta = \theta_0 < 0$ at some point on a generator of a null hypersurface then $\theta \to -\infty$ along that generator within affine parameter $2/|\theta_0|$, provided the generator extends that far and the null energy condition is satisfied. [4]

(ii) State the second law of black hole mechanics and sketch a proof. You may assume that the generators of the future horizon are complete to the future. [7]

(iii) Consider two identical Kerr black holes, with parameters (M, a), which merge to form a Schwarzschild black hole with mass M'. The fraction of the initial energy radiated in gravitational waves is $\eta = (2M - M')/(2M)$. Use the second law to derive an upper bound on η in terms of a/M. [4]

(b) Explain how to quantize a Klein-Gordon field in a globally hyperbolic spacetime. Describe why the concept of particles is ambiguous in general but well-defined in a stationary spacetime. [15]

END OF PAPER