UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 1 June, 2016 9:00 am to 12:00 pm

PAPER 310

COSMOLOGY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) In lectures we assumed homogeneity and isotropy and used the Einstein equations to obtain the Friedmann equations:

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{K}{a^2}\,, \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}\left(\rho + 3P\right)\,. \end{aligned}$$

Use them to derive the continuity equation,

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + P\right)\,.$$

The content of our universe today can be described by three fluids, radiation ($P_r = 1/3\rho_r$), matter ($P_m \approx 0$), and dark energy ($P_\Lambda = -\rho_\Lambda$). Define what we mean by the critical density ρ_{crit}, use it to define the fractional densities Ω_i, and calculate how they scale with $a(t)$

(b) Rewrite the Friedmann equations in terms of the fractional densities today to obtain

$$\begin{aligned} \dot{a}^2 &= H_0^2\left(\frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a} + \Omega_{\Lambda,0}a^2\right) - K \\ \ddot{a} &= -H_0^2\left(\frac{\Omega_{r,0}}{a^3} + \frac{\Omega_{m,0}}{2a^2} - \Omega_{\Lambda,0}a\right) \end{aligned}$$

In the absence of dark energy use the above, or otherwise, to name and describe qualitatively the evolution of a in the case of: $K > 0$, $K < 0$, and $K = 0$.

(c) In the case of $K < 0$ make this concrete by deriving the parametric solution for a universe containing only matter,

$$\begin{aligned} a(\eta) &= \frac{\Omega_{m,0}}{2\left(1-\Omega_{m,0}\right)}\left(\cosh\eta - 1\right) \\ t(\eta) &= \frac{H_0^{-1}\Omega_{m,0}}{2\left(1-\Omega_{m,0}\right)^{3/2}}\left(\sinh\eta - \eta\right) \end{aligned}$$

where η should be determined. [*You may use:*

$$\begin{aligned} \sinh^{-1} y &= \int \frac{dy}{\sqrt{1+y^2}} \\ \sinh^2\left(\frac{x}{2}\right) &= \frac{1}{2}\left(\cosh(x) - 1\right) \quad] \end{aligned}$$

2

(a) A flat universe which is both homogeneous and isotropic is described by the metric,

$$ds^2 = a^2(\tau)\left(-d\tau^2 + \delta_{ij}dx^i dx^j\right) .$$

By considering a spatial gauge transformation $\tilde{x}^i = x^i + \xi^i$ describe the gauge problem.

(b) Now, the FRW metric in flat space with only scalar perturbations has the form,

$$ds^2 = a^2(\tau)\left(-\left(1+2A\right)d\tau^2 + 2\partial_i B dx^i d\tau + \left[\left(1+2C\right)\delta_{ij} + 2\left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)E\right]dx^i dx^j\right) .$$

Consider a general scalar gauge transformation, $\tilde{x}^\mu = x^\mu + \xi^\mu$, where $\xi^0 = T(\tau, \mathbf{x})$ and $\xi^i = \partial^i L(\tau, \mathbf{x})$. Show that the metric transforms as,

$$g_{\mu\nu} = \frac{d\tilde{x}^\alpha}{dx^\mu}\frac{d\tilde{x}^\beta}{dx^\nu}\tilde{g}_{\alpha\beta} ,$$

and use this to show any **two** of the following:

$$\begin{aligned}
\widetilde{A} &= A - T' - \mathcal{H}T , \\
\widetilde{B} &= B + T - L' , \\
\widetilde{C} &= C - \mathcal{H}T - \frac{1}{3}\nabla^2 L , \\
\widetilde{E} &= E - L
\end{aligned}$$

where $\mathcal{H}$ is the comoving Hubble parameter

State why tensor perturbations are gauge invariant.

(c) One way to solve the gauge problem is to work with gauge invariant variables. A key gauge invariant variable is the constant density curvature perturbation, ζ, defined by,

$$\zeta \equiv -C + \frac{1}{3}\nabla^2 E + \mathcal{H}\frac{\delta\rho}{\bar{\rho}'} .$$

Show ζ is gauge invariant. You may use the relations above for C and E but should derive the gauge transformation relation for $\delta\rho$ by considering, or otherwise, the transformation of the scalar quantity $\rho(\tau, x^i) = \bar{\rho}(\tau) + \delta\rho(\tau, x^i)$.

One key property of ζ is that it is conserved on superhorizon scales for adiabatic perturbations. By specialising to the Newtonian gauge, where,

$$\zeta = \Phi - \frac{1}{3}\frac{\delta\rho}{\bar{\rho} + \bar{P}} ,$$

show that this is true. You may assume the following,

$$\begin{aligned}
\delta\rho' &= -3\mathcal{H}\left(\delta\rho + \delta P\right) + 3\Phi'\left(\bar{\rho} + \bar{P}\right) - \nabla\cdot\mathbf{q} , \\
\delta P_{nad} &\equiv \delta P - \frac{\bar{P}'}{\bar{\rho}'}\delta\rho , \qquad \nabla\cdot\mathbf{v} \propto \left(\frac{k}{\mathcal{H}}\right)^2 .
\end{aligned}$$

State briefly why conservation of ζ is so important in cosmology.

3

(a) Use the second law of thermodynamics, $T\mathrm{d}S = \mathrm{d}U + P\mathrm{d}V$, for an adiabatically changing volume to show that the total entropy density of a particle species is given by,

$$s = \frac{\rho + P}{T}\,, \tag{1}$$

up to an additive constant. Show explicitly that $S = Vs$ is conserved in equilibrium. For radiation, $s = c_1\, g_\star T^3$ and $\rho = c_2\, g_\star T^4$. Use equation (1) to find c_2/c_1. [*You may use* $\partial P/\partial T = (\rho + P)/T$.]

(b) A universe consists of a gas of $g_\star$ degrees of freedom at the temperature T and a massive, decoupled non-relativistic particle, ϕ, of mass m and initial abundance $Y = n_\phi/s$ which is large enough so that $\rho \approx \rho_\phi$. At the time $t = t_\mathrm{d}$, all ϕ particles instantaneously decay into radiation and quickly thermalise into the $g_\star$ degrees of freedom of the gas. Show that the temperature at the time of decay can be written as,

$$T_\mathrm{d}^3 = \frac{3M_\mathrm{Pl}^2}{c_1 g_\star Y\, m\, t_\mathrm{d}^2}\,,$$

where M_Pl denotes the reduced Planck mass. Justify any assumptions you make in reaching this result. [*You may assume that* $\Gamma = 1/t_\mathrm{d} \approx H$ *at the time of decay.*]

(c) Show that the ratio of the entropy density before and after the decay is given by,

$$\frac{s_\mathrm{after}}{s_\mathrm{before}} = c_1 \left(\frac{3}{c_2}\right)^{\frac{3}{4}} g_\star^{\frac{1}{4}} \frac{Y m \sqrt{t_d}}{\sqrt{M_\mathrm{Pl}}}\,,$$

and argue why this quantity is greater than 1. Explain this apparent contradiction with part (a).

4

A two-field model of inflation has the action,

$$S = \int \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \left(\partial_\mu \phi \partial_\nu \phi + \partial_\mu \chi \partial_\nu \chi \right) - V(\phi, \chi) \right) ,$$

with the scalar potential $V = \frac{1}{2} m_\phi^2 \phi^2 + \frac{1}{2} m_\chi^2 \chi^2$. Throughout this question you may ignore metric fluctuations and assume that spacetime is well-described by the FRW metric, $\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}\mathbf{x}^2$. The field equations are then given by,

$$\begin{aligned} -\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, g^{\mu\nu} \partial_\nu \phi \right) &= -m_\phi^2 \phi \,, \\ -\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, g^{\mu\nu} \partial_\nu \chi \right) &= -m_\chi^2 \chi \,. \end{aligned}$$

(a) Write the field equations with respect to conformal time, $\mathrm{d}\tau = 1/a\,\mathrm{d}t$, using $\mathcal{H} = a'(\tau)/a(\tau)$. Derive the equations of motion for the fluctuations $u(\tau, \mathbf{x})$ and $v(\tau, \mathbf{x})$ defined as $\phi(\tau, \mathbf{x}) = \bar{\phi}(\tau) + u(\tau, \mathbf{x})/a(\tau)$, $\chi(\tau, \mathbf{x}) = \bar{\chi}(\tau) + v(\tau, \mathbf{x})/a(\tau)$. What are the equations for the Fourier modes, $u_\mathbf{k}(\tau)$ and $v_\mathbf{k}(\tau)$? [*Recall that* $f_\mathbf{k}(\tau) = \int \frac{\mathrm{d}^3\mathbf{x}}{(2\pi)^{3/2}} \exp(-i\mathbf{k}\cdot\mathbf{x}) f(\tau, \mathbf{x})$.]

(b) Assume $H^2 < m_\chi^2 < 2H^2$ and take H to be approximately constant during inflation. Show that on superhorizon scales the equation for $v_\mathbf{k}(\tau)$ is solved by $v_\mathbf{k} = c\,(k\tau)^{\frac{1}{2} \pm \nu}/\sqrt{2k}$ for some constant c, and determine ν. Are the corresponding solutions for $\chi_\mathbf{k}$ constant, growing, or decaying as $\tau \to 0$? [*Take* $a(\tau) \approx -1/(H\tau)$ *for* $\tau < 0$.]

(c) The quantum Heisenberg picture operator $\hat{v}_\mathbf{k}$ has the mode expansion,

$$\hat{v}_\mathbf{k} = v_\mathbf{k} \hat{a}_\mathbf{k} + v_\mathbf{k}^* \hat{a}_\mathbf{k}^\dagger \,,$$

where $[\hat{a}_\mathbf{k}, \hat{a}_\mathbf{k}^\dagger] = \delta^3(\mathbf{k} - \mathbf{k}')$. Compute the variance of the field fluctuation $\langle \hat{v}(\tau, \mathbf{x} = \mathbf{0}), \hat{v}^\dagger(\tau, \mathbf{x} = \mathbf{0}) \rangle$ in terms of the dimensionless power spectrum of v, $\Delta_v \equiv (k^3/2\pi^2)|v_\mathbf{k}|^2$ and show that the dimensionless power spectrum of $\delta\chi = v/a$ is given by,

$$\Delta_{\delta\chi}(k) = |c|^2 \left(\frac{H}{2\pi} \right)^2 \left(\frac{k}{aH} \right)^{3-2\nu} .$$

END OF PAPER