UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 30 May, 2016 9:00 am to 12:00 pm

PAPER 309

GENERAL RELATIVITY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $(\mathcal{M}, g)$ be a spacetime and Σ a spatial hypersurface with future pointing, timelike unit normal related in adapted coordinates to the lapse function α and the shift vector β^i $(i = 1,\, 2,\, 3)$ by $n^\mu = [\alpha^{-1},\, -\beta^i\alpha^{-1}]$.

(i) Give the definition of the spatial projection operator $\perp^\mu{}_\alpha$, of the spatial projection $(\perp T)_{\alpha\beta}$ of a rank $(0, 2)$ tensor $T_{\alpha\beta}$ and show that

$$(\perp T)_{\alpha\beta} n^\alpha = (\perp T)_{\alpha\beta} n^\beta = 0\,.$$

(ii) The totally antisymmetric tensor $\epsilon_{\alpha\beta\gamma\delta}$ on $(\mathcal{M}, g)$ is defined through $\epsilon_{0123} = |g|^{1/2}$, acquiring a factor -1 for any exchange of two indices and $\epsilon_{\alpha\beta\gamma\delta} = 0$ if two or more indices are equal. Here, $g \equiv \det g_{\alpha\beta}$. Let γ_{ij} denote the spatial metric and $\gamma \equiv \det \gamma_{ij}$ its determinant. Use the relation $g = -\alpha^2\gamma$ to show that in coordinates adapted to the $3+1$ split, the 3-dimensional totally antisymmetric tensor $\tilde{\epsilon}_{ijk}$ on (Σ, γ) is given by

$$\tilde{\epsilon}_{ijk} = \epsilon_{\mu ijk} n^\mu\,.$$

(iii) The Weyl tensor on $(\mathcal{M}, g)$ is defined in terms of the Riemann tensor $R_{\alpha\beta\gamma\delta}$, the Ricci tensor $R_{\alpha\beta}$ and the Ricci scalar R by

$$C_{\alpha\beta\gamma\delta} \equiv R_{\alpha\beta\gamma\delta} + g_{\alpha[\delta}R_{\gamma]\beta} + g_{\beta[\gamma}R_{\delta]\alpha} + \frac{1}{3} R\, g_{\alpha[\gamma}g_{\delta]\beta}\,.$$

Its electric and magnetic part are defined as

$$\begin{aligned} E_{\alpha\beta} &\equiv C_{\alpha\rho\beta\sigma} n^\rho n^\sigma\,, \\ B_{\alpha\beta} &\equiv \frac{1}{2}\epsilon_{\alpha\lambda\mu\nu} C^{\mu\nu}{}_{\beta\rho} n^\lambda n^\rho\,. \end{aligned}$$

The Gauss-Codazzi equations are given by

$$\begin{aligned} (\perp R)_{\alpha\beta\gamma\delta} &= \mathcal{R}_{\alpha\beta\gamma\delta} + 2K_{\alpha[\gamma}K_{\delta]\beta}\,, \\ \perp(R_{\alpha\beta\gamma\delta})n^\delta &= -D_\alpha K_{\beta\gamma} + D_\beta K_{\alpha\gamma}\,, \end{aligned}$$

where D_α and $\mathcal{R}_{\alpha\beta\gamma\delta}$ are the spatial covariant derivative and the Riemann tensor associated with $\gamma_{\alpha\beta}$.

1) How are the Weyl and Riemann tensor related in a vacuum spacetime?
2) Show that in a vacuum spacetime on Σ, $E_{\alpha\beta} = (\perp E)_{\alpha\beta}$ and $B_{\alpha\beta} = (\perp B)_{\alpha\beta}$.
3) Show that on a spatial hypersurface Σ of a vaccum spacetime $(\mathcal{M}, g)$,

$$\begin{aligned} E_{\alpha\beta} &= k_1\, \mathcal{R}_{\alpha\beta} + k_2\, K K_{\alpha\beta} + k_3\, K^\nu{}_\beta K_{\nu\alpha}\,, \\ B_{\alpha\beta} &= k_4\, \tilde{\epsilon}_\alpha{}^{\mu\nu} D_\mu K_{\nu\beta}\,, \end{aligned}$$

where k_1, k_2, k_3 and k_4 are real constants that you should calculate.

2

(i) Derive the geodesic equation in coordinate form from the defining equation $\nabla_{\boldsymbol{X}}\boldsymbol{X} = 0$ of an affinely parametrized geodesic.

(ii) Consider the Vaidya metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{2M(v)}{r}\right)dv^2 + 2dv\,dr + r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2\,.$$

Use the Euler-Lagrange equations of the Lagrangian

$$L = -g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\,,$$

where τ denotes proper time, to calculate all non-vanishing Christoffel symbols for the Vaydia metric.

(iii) Write down the r and v components of the geodesic equation for the special case of radial geodesics where $\theta = \text{const}$, $\phi = \text{const}$. Show that (1) curves satisfying $v = \text{const}$, are radial null geodesics and (2) that curves satisfying

$$\frac{dr}{dv} = \frac{1}{2}\left(1 - \frac{2M(v)}{r}\right)\,, \qquad (\dagger)$$

are radial null geodesics.

(iv) Consider radial null geodesics satisfying Eq. (†). Show that if such a geodesic is given by $r = r_0(v)$, the function $M(v)$ must be

$$M(v) = \frac{1}{2}r_0(v)\left[1 - 2\dot{r}_0(v)\right]\,,$$

where the dot denotes d/dv.
Consider the special case $r_0(v) = \beta w^n$, where $\beta > 0$ and $n > 1$ are constants and

$$v = 2\beta w^n + w\,, \qquad v \geqslant 0\,, \quad w \geqslant 0\,.$$

Show that in the limit of small v, to leading order

$$M(v) = c_1 v^n\,.$$

Determine the constant c_1.

3

(a) Let $\mathcal{M}$ be a manifold, $p \in \mathcal{M}$ and $\boldsymbol{A}$, $\boldsymbol{B}$ rank (1,1) tensors which define maps $A : \mathcal{T}_p(\mathcal{M}) \to \mathcal{T}_p(\mathcal{M})$, $B : \mathcal{T}_p(\mathcal{M}) \to \mathcal{T}_p(\mathcal{M})$, with

$$\boldsymbol{A}(\boldsymbol{X}) : \mathcal{T}_p^*(\mathcal{M}) \to \mathbb{R}\,, \quad \boldsymbol{\eta} \mapsto \boldsymbol{A}(\boldsymbol{X}, \boldsymbol{\eta}) \quad \text{for all } \boldsymbol{X} \in \mathcal{T}_p(\mathcal{M})\,,$$

$$\boldsymbol{B}(\boldsymbol{X}) : \mathcal{T}_p^*(\mathcal{M}) \to \mathbb{R}\,, \quad \boldsymbol{\eta} \mapsto \boldsymbol{B}(\boldsymbol{X}, \boldsymbol{\eta}) \quad \text{for all } \boldsymbol{X} \in \mathcal{T}_p(\mathcal{M})\,.$$

Define

$$\boldsymbol{C} : \mathcal{T}_p(\mathcal{M}) \to \mathcal{T}_p(\mathcal{M})\,, \qquad \boldsymbol{X} \mapsto \boldsymbol{C}(\boldsymbol{X}) \equiv \boldsymbol{B}(\boldsymbol{A}(\boldsymbol{X}))\,.$$

Show that $\boldsymbol{C}$ is a tensor of rank (1,1), i.e. a multilinear map $C : \mathcal{T}_p(\mathcal{M}) \times \mathcal{T}_p^*(\mathcal{M}) \to \mathbb{R}$, and determine the components ${C_\alpha}^\beta$ of $\boldsymbol{C}$ in terms of the components of $\boldsymbol{A}$ and $\boldsymbol{B}$.

(b) Let $\mathcal{M}$ be a manifold, $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{Z}$ smooth vector fields on $\mathcal{M}$, $f : \mathcal{M} \to \mathbb{R}$ a smooth function and let $\mathcal{L}_{\boldsymbol{X}}$ denote the Lie derivative along the vector field $\boldsymbol{X}$.

(i) Give index free expressions for the result of the Lie derivative acting on a function and the Lie derivative acting on a vector field, i.e. for $\mathcal{L}_{\boldsymbol{X}} f$ and $\mathcal{L}_{\boldsymbol{X}} \boldsymbol{Y}$.

(ii) Show that the Lie derivative acting on 1) functions and 2) on vector fields satisfies

$$\big[\mathcal{L}_{\boldsymbol{X}}, \mathcal{L}_{\boldsymbol{Y}}\big] = \mathcal{L}_{[\boldsymbol{X},\boldsymbol{Y}]}\,,$$

and

$$\big[[\mathcal{L}_{\boldsymbol{X}}, \mathcal{L}_{\boldsymbol{Y}}], \mathcal{L}_{\boldsymbol{Z}}\big] + \big[[\mathcal{L}_{\boldsymbol{Y}}, \mathcal{L}_{\boldsymbol{Z}}], \mathcal{L}_{\boldsymbol{X}}\big] + \big[[\mathcal{L}_{\boldsymbol{Z}}, \mathcal{L}_{\boldsymbol{X}}], \mathcal{L}_{\boldsymbol{Y}}\big] = 0\,,$$

where brackets denote the commutator, i.e. $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X$.

4

(i) Let $(\mathcal{M}, \Gamma)$ be a manifold with a torsion free connection Γ and a one parameter family of geodesics $\gamma : I \times I' \to \mathcal{M},\ \ I, I' \subset \mathbb{R},\ \ (s,t) \mapsto \gamma(s,t)$. Let $\boldsymbol{T}$ be the tangent vector to the geodesics $\gamma(s = \text{const}, t)$ and $\boldsymbol{S}$ tangent to the curves $\gamma(s, t = \text{const})$. Use the definition of the Riemann tensor, $\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z} = \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\boldsymbol{Z} - \nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}\boldsymbol{Z} - \nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\boldsymbol{Z}$ for vector fields $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{Z}$, to derive the equation of geodesic deviation

$$\nabla_{\boldsymbol{T}}\nabla_{\boldsymbol{T}}\boldsymbol{S} = \boldsymbol{R}(\boldsymbol{T},\boldsymbol{S})\boldsymbol{T} = R^{\alpha}{}_{\beta\gamma\delta}T^{\beta}T^{\gamma}S^{\delta}\boldsymbol{e}_{\alpha}\,,$$

where at each point of the geodesics, $\{\boldsymbol{e}_\alpha\}$ is a basis of the tangent space.

(ii) In the weak field limit the metric is given by a perturbation $h_{\mu\nu}$ of order $\mathcal{O}(\epsilon),\ \epsilon \ll 1$, on a Minkowski background $\eta_{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$: $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$. To order ϵ, the Levi Civita connection and Riemann tensor are

$$\begin{aligned}\Gamma^{\mu}_{\nu\rho} &= \frac{1}{2}\eta^{\mu\sigma}(\partial_\rho h_{\sigma\nu} + \partial_\nu h_{\rho\sigma} - \partial_\sigma h_{\nu\rho})\,,\\ R_{\mu\nu\rho\sigma} &= \frac{1}{2}(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma} - \partial_\sigma\partial_\nu h_{\mu\rho})\,.\end{aligned}$$

Show that the components of the Riemann tensor are invariant under gauge transformations $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu V_\nu + \partial_\nu V_\mu$, where V_μ is of order $\mathcal{O}(\epsilon)$.

(iii) Consider two test particles initially at rest in an inertial frame in Minkowski spacetime with a separation vector L^μ. Under passage of a gravitational wave, represented by a metric perturbation $h_{\mu\nu}$ of order $\mathcal{O}(\epsilon)$, the separation vector is allowed to vary by a small ξ^μ, i.e. is given by $L^\mu + \xi^\mu$ with $L^\mu = \text{const}$ and $\xi^\mu = \mathcal{O}(\epsilon)$ may depend on time. Determine the equation of geodesic deviation to linear order in the transverse traceless gauge ($h \equiv \eta^{\mu\nu}h_{\mu\nu} = 0,\ \partial^\nu h_{\mu\nu} = 0$) for the case of a planar gravitational wave propagating in the x^3 direction, given in this gauge by

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_\times & 0 \\ 0 & H_\times & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{ik_\mu x^\mu}\,, \qquad k_\mu = \text{const}\,.$$

Expand for this calculation the connection coefficients and Riemann tensor in terms of the $h_{\mu\nu}$ and write your final result in the form $\ddot{\xi}^\mu = \ldots$, where a dot denotes a derivative with respect to coordinate time $t = x^0$.

(iv) The quadrupole formula gives the energy flux of a gravitational wave in terms of the quadrupole tensor I_{ij} of its source as

$$\left\langle \frac{dE}{dt} \right\rangle = \frac{G}{5c^5}\left\langle \dddot{Q}_{ij}\dddot{Q}_{ij} \right\rangle\,, \qquad Q_{ij} = I_{ij} - \frac{1}{3}I_{kk}\delta_{ij}\,, \qquad I_{ij} = \int T_{00}y^i y^j d^3y\,.$$

Use this formula to compute $\langle dE/dt \rangle$ sourced by a point mass m harmonically oscillating along the x^3 axis with frequency ω and amplitude L.

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