UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 7 June, 2016 9:00 am to 11:00 am

PAPER 308

CLASSICAL AND QUANTUM SOLITONS

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider the Lagrangian density of ϕ^4-theory in $1+1$ dimensions,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\left(1-\phi^2\right)^2 .$$

Find the field equation and show that $\phi(x) = \tanh(x-a)$ is a kink solution for any a. Sketch a graph of the kink solution, and explain why the kink is topologically stable.

Let $\phi(x,t)$ be a general, finite-energy solution of the field equation that approaches the vacuum $\phi = \pm 1$ as $x \to \pm\infty$. Derive an expression for the momentum density $\mathcal{P}$ of the field, and show that, for fixed b,

$$\frac{d}{dt}\int_{-\infty}^{b} \mathcal{P}dx$$

can be expressed in terms of the field and its first derivatives evaluated at b. Interpret your result in terms of a force.

An initial kink-antikink configuration of the form

$$\phi(x) = \tanh(x+c) - \tanh(x-c) - 1$$

is given at $t=0$, where $c \gg 0$. Calculate the force exerted on the kink by the antikink.

2

Describe two methods by which the degree of a smooth map $g : S^2 \to S^2$ can be calculated. Explain in outline why your two methods give the same answer. What is the degree of a rational map

$$R(z) = \frac{p(z)}{q(z)}$$

where $z \in \mathbb{C}\cup\{\infty\}$ is a Riemann sphere coordinate and p, q are polynomials of (maximum) algebraic degree k? Justify your answer.

Describe how rational maps can be used to find approximate Skyrmions, and briefly mention their advantages and limitations.

Discuss the symmetry properties of the rational maps

(i) $R(z) = z^n \quad (n = 1, 2, 3, \dots)$,

(ii) $R(z) = \dfrac{z^4 + 2\sqrt{3}iz^2 + 1}{z^4 - 2\sqrt{3}iz^2 + 1}$,

and explain how useful these rational maps are in the context of Skyrmions.

3

Write an essay on nuclei, Skyrmions and QCD, and the relationships between them.

END OF PAPER