UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 2 June, 2016 9:00 am to 12:00 pm

PAPER 306

STRING THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) A unit mass non-relativistic point particle on the x axis has the action

$$I = \int d\lambda \left\{ \dot{x}p - \dot{t}E + e\left[E - H(x,p)\right]\right\} , \quad H(x,p) = \frac{1}{2}p^2 + \frac{g}{x^2} ,$$

where $e(\lambda)$ is a Lagrange multiplier, g a constant, and an overdot means $d/d\lambda$. Write down the equations of motion. Write down the canonical Poisson Bracket relations. What is the Noether charge associated to the symmetry $t \to t+$constant? Let $D = tE - \frac{1}{2}xp$ and verify that D is a constant of the motion. Find the infinitesimal symmetry transformations that it generates. What is the Poisson bracket of D with E?

Now consider the following variations of the canonical variables for infinitesimal parameter $\beta(\lambda)$:

$$\delta_\beta t = -t^2\beta \, , \quad \delta_\beta x = -tx\beta \, , \quad \delta_\beta E = (2tE - xp)\beta \, , \quad \delta_\beta p = (tp - x)\beta \, .$$

Show that $\delta_\beta(E - H) = 2t(E - H)\beta$. Now find a variation $\delta_\beta e$ such that

$$\delta_\beta I = \int d\lambda \left\{ \dot{\beta}K + \frac{d}{d\lambda}(\ldots)\right\}$$

for a function K that you should also find. Verify that the equations of motion imply $\dot{K} = 0$, and compute the Poisson brackets of K with E and D. Comment briefly on your result.

(ii) The Nambu-Goto action for an open string in a D-dimensional Minkowski spacetime has a phase-space action of the form

$$I[X,P;e,u] = \int dt \int_0^\pi d\sigma \left\{ \dot{X}^m P_m - \frac{1}{2}e\left(P^2 + (TX')^2\right) - u\, X'^m P_m \right\} .$$

Write down the equations of motion, and use them to determine the condition(s) that are required for the total D-momentum to be a constant of the motion; i.e. for $\dot{\mathcal{P}} = 0$, where

$$\mathcal{P}_m = \int_0^\pi d\sigma\, P_m \, .$$

Use your result to show that $\dot{\mathcal{P}} = 0$ for an open string with free ends.

Is $\mathcal{P}_1$ a constant of the motion for a string with endpoints fixed to the hyperplane $x^1 = 0$? If not, what happens to any momentum that escapes from the string?

2

(i) In light-cone coordinates, $\{x^+, x^-, x^I\}$, the metric of D-dimensional Minkowski space is $ds^2 = 2dx^+dx^- + dx^I dx^I$. How are these coordinates related to the standard cartesian coordinates x^m?

The antisymmetric tensor field B_{mn}, on D-dimensional Minkowski space, satisfies the equation

$$\Box_D B_{mn} + \partial_m \partial^p B_{np} - \partial_n \partial^p B_{mp} = 0\,.$$

Verify that this equation is unchanged by the gauge transformation $B_{mn} \to B_{mn} + 2\partial_{[m}\alpha_{n]}$, for arbitrary vector field α_n. Assuming that ∂_- is invertible, show that the gauge invariance is eliminated by the gauge fixing condition $B_{-m} = 0$ $(m = +, I)$. What are the independent components of B_{mn} in this gauge, and what equation do they satisfy.

Explain *briefly* how your results are relevant to the spectrum of the closed Nambu-Goto string. How can the string interact with a background antisymmetric tensor field B_{mn} in a way that is invariant under a gauge transformation of B_{mn}.

(ii) An *infinite* Nambu-Goto string of unit tension, in a three-dimensional Minkowski spacetime with cartesian coordinates $\{X^m; m = 0, 1, 2\}$, has the phase-space action

$$I = \int dt \int_{-\infty}^{\infty} d\sigma \left\{ \dot{X}^m P_m - \frac{1}{2} e \left[P^2 + (X')^2\right] - u\left[(X^m)' P_m\right] \right\}\,.$$

By imposing the Monge gauge

$$X^0 = t\,, \qquad X^1 = \sigma\,,$$

show that the constraints can be solved for P_0 and P_1 in terms of $X^2 \equiv \Phi(t,\sigma)$ and $P_2 \equiv \Pi(t,\sigma)$. Hence show that the canonical Hamiltonian in Monge gauge takes the form

$$H[\Phi, \Pi] = \int_{-\infty}^{\infty} d\sigma\, \sqrt{F(\Phi')F(\Pi)}\,,$$

for a function F that you should find (you may assume that $H > 0$). Given a solution of the Monge-gauge equations of motions that satisfies $\Pi = \Phi'$, show that it also satisfies $\dot{\Phi} = \Phi'$. Interpret the particular solution $\Phi = k(\sigma + t)$ and $\Pi = k$ for constant k. Comment on the $k > 1$ case.

3

(i) Write down the "quantum" action $I[X, b, c, \tilde{b}, \tilde{c}]$ for the closed NG string in conformal gauge including the Faddeev-Popov ghost action. Write down the commutator algebra of the Noether charges $\{\mathcal{L}_n, \tilde{\mathcal{L}}_n; n \in \mathbb{Z}\}$ associated to conformal invariance of $I[X, b, c, \tilde{b}, \tilde{c}]$, and explain *without calculation* how this allows a determination of the critical dimension.

(ii) Write an essay on the path-integral quantization of the Nambu-Goto string, with the emphasis on features that lead to improved ultra-violet behaviour when compared to General Relativity quantized as a Minkowski space interacting spin-2 QFT. You should cover the following points:

- How the sum-over-worldsheets approach to the scattering of strings (e.g. those in states representing gravitons) eliminates the point-like interaction vertices of Feynman diagrams (and why this is relevant to UV divergences).
- How, in the Euclidean path integral, the sum over worldsheets reduces to a sum over Riemann surfaces, leading to a string-loop expansion analogous to the Feynman diagram loop expansion of QFT.
- How the string spectrum leads to a modification of the gravitational force at the string scale. You should illustrate this point by stating, and then using, properties of the Virasoro amplitude

$$A(s,t) \propto \frac{\Gamma(-1-t)\Gamma(-1-s)\Gamma(-1-u)}{\Gamma(u+2)\Gamma(s+2)\Gamma(t+2)} \qquad (u = -4 - s - t).$$

- The relevance of the modular group of Riemann surfaces, in particular the torus, in the context of the string-loop expansion of scattering amplitudes.

4

(i) A particle in a Minkowski spacetime has the action

$$I[x,p,\psi;e,\chi] = \int dt \left\{ \dot{x}^m p_m + \frac{i}{2}\psi^m \dot{\psi}^n \eta_{mn} - e\,\mathcal{H}(x,p,\psi) - i\chi\,\mathcal{Q}(x,p,\psi) \right\} ,$$

where $\psi^m(t)$ are *anti-commuting*, as is the Lagrange multiplier $\chi(t)$. Write down the canonical Poisson bracket relations and use them to find the Poisson bracket relations of the particular constraint functions

$$\mathcal{Q} = \psi^m p_m \,, \qquad \mathcal{H} = \frac{1}{2}p^2 \,.$$

Hence show that the physical-state conditions of the quantum theory are equivalent to the massless Dirac equation.

(ii) The open Ramond string in ten-dimensional Minkowski space has the following light-cone gauge action:

$$I_R = \int dt \left\{ \dot{x}^m p_m + \frac{i}{2}\mathbf{d}_0 \cdot \mathbf{d}_0 + \sum_{k=1}^{\infty} \left(\frac{i}{k}\boldsymbol{\alpha}_{-k} \cdot \dot{\boldsymbol{\alpha}}_k + i\mathbf{d}_{-k} \cdot \dot{\mathbf{d}}_k \right) - \frac{1}{2}e_0\left(p^2 + 2\pi T N_R\right) \right\} ,$$

where the transverse oscillator variables $(\boldsymbol{\alpha}_k, \mathbf{d}_k)$ are 8-vectors. Write down an expression for the Ramond level number N_R in terms of these variables. Write down the canonical (anti)commutation relations for the operators $\hat{\boldsymbol{\alpha}}_k$ and $\hat{\mathbf{d}}_k$ ($k \in \mathbb{Z}$). An oscillator vacuum $|0\rangle$ is a state annihilated by both $\hat{\boldsymbol{\alpha}}_k$ and $\hat{\mathbf{d}}_k$ for $k > 0$. Given an operator ordering such that $\hat{N}_R|0\rangle = 0$, show that the eigenvalues N of $\hat{N}_R$ are non-negative integers. Why should you expect $N = 0$ states to be massless?

Let $\hat{d}_0^I$ ($I = 1,\ldots,8$) be the components of $\hat{\mathbf{d}}_0$. What are the anticommutation relations satisfied by the hermitian operators $\gamma^I = \sqrt{2}\,\hat{d}_0^I$? Given that $|0\rangle$ is an oscillator vacuum, show that the eight states $\gamma^I|0\rangle$ are also oscillator vacua. Show further that these eight states are linearly independent over $\mathbb{R}$. Similarly, using that $\gamma^1|0\rangle$ is an oscillator vacuum, show that the eight states $\gamma^I\gamma^1|0\rangle$ are also oscillator vacua, again linearly independent over $\mathbb{R}$; do they include $|0\rangle$?

Show that $\gamma_9 = \gamma_1\gamma_2\cdots\gamma_8$ is hermitian. Show too that γ_9^2 is the identity operator. Given that $|0\rangle$ is an eigenstate of γ_9 with eigenvalue 1 (positive chirality) show that $\gamma^I|0\rangle$ are eigenstates with eigenvalue -1 (negative chirality). Show too that $\gamma^I\gamma^1|0\rangle$ are positive chirality states. Finally, show that any positive chirality state is orthogonal to any negative chirality state and hence deduce that there are at least 16 independent oscillator vacua.

END OF PAPER