UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 1 June, 2016 1:30 pm to 4:30 pm

PAPER 305

THE STANDARD MODEL

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Recall the expansion for a Dirac field $\psi(x)$ with mass m,

$$\psi(x) = \sum_{p,s} \left[b^s(p) u^s(p) e^{-ip\cdot x} + d^{s\dagger}(p) v^s(p) e^{ip\cdot x} \right] ,$$

where $(\not{p} - m)u^s(p) = 0$, $(\not{p} + m)v^s(p) = 0$ and $s = \pm\frac{1}{2}$. Explain the meaning of b^s, $d^{s\dagger}$, u^s and v^s. Assuming the results,

$$u^s(p_P) = \gamma^0 u^s(p)\,, \qquad v^s(p_P) = -\gamma^0 v^s(p)\,,$$

$$\hat{P} b^s(p) \hat{P}^{-1} = \eta_P b^s(p_P)\,, \qquad \hat{P} d^{s\dagger}(p) \hat{P}^{-1} = -\eta_P d^{s\dagger}(p_P)\,,$$

show that under a parity transformation (P),

$$\psi(x) \mapsto \hat{P}\psi(x)\hat{P}^{-1} = \eta_P \gamma^0 \psi(x_P)\,,$$

$$\bar{\psi}(x) \mapsto \hat{P}\bar{\psi}(x)\hat{P}^{-1} = \eta_P^* \bar{\psi}(x_P)\gamma^0\,,$$

where $|\eta_P| = 1$, $x_P^\mu = (x^0, -\mathbf{x})$ and $p_P^\mu = (p^0, -\mathbf{p})$.

(b) Under a charge-conjugation transformation (C),

$$\psi(x) \mapsto \eta_C C \bar{\psi}^T\,,$$

where $\gamma^{\mu\,T} = -C\gamma^\mu C^{-1}$, $|\eta_C| = 1$ and you may assume that $\gamma^{0\,T} = \gamma^0$. Given that $\bar{\psi}(x)\psi(x)$ is invariant under C, find a constraint on C. How do $\bar{\psi}(x)\gamma^\mu\psi(x)$ and $\bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)$ transform under P and C? [Here $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$ and you may assume that $\gamma^{5T} = C\gamma^5 C^{-1}$ and $C^{-1} = C^T$.]

(c) Now consider an $SU(3)$ gauge theory where $\psi(x)$ is in the fundamental representation, with generators denoted by T^a, and is coupled to the gauge bosons A_μ^a $(a = 1, \ldots 8)$. Given that the relevant term in the Lagrangian is, $ig\bar{\psi}A_\mu^a T^a \gamma^\mu \psi$, and that this leads to an interaction that is invariant under P and C, derive expressions for the transformation of $A_\mu^a T^a$ under P and C. [*Hint: Recall that* T^a *are matrices.*]

Suppose that the following term is added to the Lagrangian,

$$\mathcal{L}_\theta(x) = \theta\, \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a(x) F_{\rho\sigma}^a(x)\,,$$

where θ is a real constant and,

$$F_{\mu\nu}^a T^a = \partial_\mu A_\nu^a T^a - \partial_\nu A_\mu^a T^a + ig[A_\mu^b T^b, A_\nu^c T^c]\,.$$

Determine how $F_{\mu\nu}^a T^a$ transforms under CP and hence whether or not $\mathcal{L}_\theta(x)$ leads to an interaction that conserves CP.

(d) Give an example of a process where CP violation has been observed experimentally. What is the minimum number of quark generations required for CP violation to be possible in the Standard Model?

2

(a) Consider a field theory for an n-component real scalar field ϕ_i $(i = 1, \ldots n)$ with a Lagrangian that is invariant under global $SO(n)$ transformations,

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi_i \partial_\mu \phi_i - \frac{1}{2}m^2 \phi_i \phi_i - \frac{\lambda}{4}(\phi_i \phi_i)^2\,, \qquad \lambda > 0\,.$$

For both (i) $m^2 > 0$ and (ii) $m^2 < 0$, identify the relevant physical degrees of freedom and their masses (ignoring any quantum corrections). For case (ii) explain how the symmetry is spontaneously broken and identify the unbroken symmetry.

(b) Now consider an $SU(2)$ gauge theory involving a 3-component real scalar field ϕ_i and a gauge field B^a_μ $(i, a = 1, \ldots 3)$ with Lagrangian,

$$\mathcal{L} = \frac{1}{2}(D_\mu \phi)_i (D^\mu \phi)_i - \frac{1}{2}m^2 \phi_i \phi_i - \frac{\lambda}{4}(\phi_i \phi_i)^2 - \frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu}\,, \quad \lambda > 0, \quad m^2 < 0\,,$$

where $D_\mu = \partial_\mu + igt^a B^a_\mu$, $(t^a)_{jk} = -i\epsilon_{ajk}$ and $F^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu - g\epsilon^{abc}B^b_\mu B^c_\nu$. Why, without loss of generality, can the fluctuations of ϕ about the minimum be taken to be $\phi(x) = (0, 0, v + \eta(x))^T$ where $\eta(x)$ is real? Discuss how the symmetry is spontaneously broken by the vacuum, identify the unbroken symmetry and write the Lagrangian in terms of physical fields. Give the masses of the physical fields (ignoring any quantum corrections) and briefly summarize their interactions.

Explain briefly in what ways this theory, after adding couplings to fermions, could be a suitable description of weak and electromagnetic interactions. In what crucial respects does it differ from the electroweak part of the Standard Model?

3

(a) In the Standard Model, the part of the Lagrangian that describes the coupling of W-bosons to quarks and antiquarks is,

$$\mathcal{L}_I = \frac{g}{2\sqrt{2}} \left(J^\mu W_\mu^+ + J^{\mu\dagger} W_\mu^- \right)$$

where,

$$J^\mu = \sum_{q \in \{u,c,t\}} \sum_{q' \in \{d,s,b\}} V_{qq'}\, \bar{q}\gamma^\mu(1-\gamma^5)q' \,.$$

Briefly describe how the CKM matrix elements $V_{qq'}$ arise. How many independent real parameters are there in the CKM matrix in the Standard Model (with 3 generations of quarks)? How many would there be in the analogous matrix if there were 4 generations of quarks?

(b) Draw the tree-level Feynman diagram for the decay $W^+ \to q\bar{q}'$ where $q \in \{u,c,t\}$ and $\bar{q}' \in \{\bar{d}, \bar{s}, \bar{b}\}$.

Show that the decay rate for an unpolarised W^+ to a quark-antiquark pair $(q\bar{q}')$ is,

$$\Gamma_{W^+ \to q\bar{q}'} = \frac{G_F}{\sqrt{2}} \frac{M_W^3}{6\pi} |V_{qq'}|^2 \,,$$

where the masses of q and $\bar{q}'$ and the strong interactions between them (i.e. hadronization effects) are neglected, and $\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2}$. [*Hint: The following expressions may be used without proof:*

$$\langle 0 \,|W_\mu^+|\, W^+(p)\rangle = \epsilon_\mu(p) \,, \qquad \sum_{W\ spins} \epsilon_\mu(p)\epsilon_\nu(p)^* = -g_{\mu\nu} + \frac{p_\mu p_\nu}{M_W^2} \,,$$

$$\begin{aligned} Tr(\gamma^{\mu_1}\dots\gamma^{\mu_n}) &= 0 \quad for\ n\ odd \,, \\ Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \,, \\ Tr(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4i\epsilon^{\mu\nu\rho\sigma} \,, \end{aligned}$$

and the decay rate for $A(p) \to B(k) + C(k')$ is,

$$\Gamma_{A\to BC} = \frac{1}{2m} \int \frac{d^3k}{(2\pi)^3 2k^0} \int \frac{d^3k'}{(2\pi)^3 2k'^0} (2\pi)^4 \delta^{(4)}(p-k-k') |\mathcal{M}|^2 \,,$$

where particle A has mass m.]

List all possible combinations of q and $\bar{q}'$ to which the W^+ boson can decay in the Standard Model. Calculate the total decay rate to all these combinations.

(c) In the limit of massless leptons, why is the decay rate for $W^+ \to e^+\nu_e$ nonzero whereas the decay rate for $\pi^+ \to e^+\nu_e$ vanishes?

4

(a) In the limit that neutrinos are massless, briefly describe the differences between the interactions of electrons and the interactions of electron neutrinos in the Standard Model. What is the experimental evidence that at least some flavours of neutrinos have non-zero mass?

(b) Draw tree-level Feynman diagrams for the scattering processes $\nu_e + d \to e^- + u$ and $\nu_e + \bar{u} \to e^- + \bar{d}$. Briefly explain why, at energy scales far below M_W, the effective Lagrangian relevant for these processes is,

$$\mathcal{L}_{\text{eff}} = -\frac{G_F}{\sqrt{2}}\left[\bar{u}_e \gamma^\alpha (1-\gamma^5) u_\nu\right]\left[\bar{u}_u \gamma_\alpha (1-\gamma^5) u_d + \bar{v}_u \gamma_\alpha (1-\gamma^5) v_d\right],$$

where $\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2}$. [Throughout this question the mixing between different generations and the masses of leptons, neutrinos and quarks can be neglected. You may also assume that energies are such that $\mathcal{L}_{\text{eff}}$ can be used.] Show that,

$$\begin{aligned}
\frac{d\sigma_d(k)}{dy_q} \equiv \frac{d\sigma}{dy_q}\Big(\nu_e(p) + d(k) \to e^-(p') + u(k')\Big) &= G_F^2\, A(s)\,, \\
\frac{d\sigma_{\bar{u}}(k)}{dy_q} \equiv \frac{d\sigma}{dy_q}\Big(\nu_e(p) + \bar{u}(k) \to e^-(p') + \bar{d}(k')\Big) &= G_F^2\, B(s, y_q)\,,
\end{aligned}$$

where $A(s)$ and $B(s, y_q)$ are functions which you should find, $y_q = \frac{k\cdot q}{k\cdot p}$, $s = (p+k)^2$ and $q = p - p' = k' - k$. [*Hint: The following expressions may be used without proof:*

$$\begin{aligned}
Tr(\gamma^{\mu_1}\dots\gamma^{\mu_n}) &= 0 \quad \textit{for } n \textit{ odd}\,, \\
Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})\,, \\
Tr(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4i\epsilon^{\mu\nu\rho\sigma}\,, \\
\epsilon^{\alpha\beta\sigma\rho}\epsilon_{\alpha\beta\lambda\tau} &= -2(\delta^\sigma_\lambda\delta^\rho_\tau - \delta^\sigma_\tau\delta^\rho_\lambda)\,,
\end{aligned}$$

and the differential cross section for $A(p_A) + B(p_B) \to C(p_C) + D(p_D)$ *is,*

$$d\sigma = \frac{1}{|\vec{v}_A - \vec{v}_B|}\frac{1}{4p_A^0 p_B^0}\left(\frac{d^3p_C}{(2\pi)^3 2p_C^0}\right)\left(\frac{d^3p_D}{(2\pi)^3 2p_D^0}\right)(2\pi)^4\delta^{(4)}(p_A + p_B - p_C - p_D)|\mathcal{M}|^2\,.]$$

(c) Now consider the deep inelastic scattering of a neutrino off a hadron H containing up and down quarks and antiquarks. In the parton model, working in a frame where the hadron's mass can be neglected,

$$\frac{d\sigma_H}{dy} \equiv \frac{d\sigma}{dy}\Big(\nu_e(p) + H(P_H) \to e^-(p') + X(P_X)\Big) = \int_0^1 \left[\frac{d\sigma_d(\xi P_H)}{dy} q_d(\xi) + \frac{d\sigma_{\bar{u}}(\xi P_H)}{dy} q_{\bar{u}}(\xi)\right] d\xi\,,$$

where the momentum of the interacting quark/antiquark is $k = \xi P_H$ and $q_d(\xi)$, $q_{\bar{u}}(\xi)$ are parton distribution functions (PDFs). Show that, $\xi = \frac{-q^2}{2P_H\cdot q} \equiv x$ and $y \equiv \frac{P_H\cdot q}{P_H\cdot p} = y_q$. [*Hint: start by expanding out* $(\xi P_H + q)^2 = 0$.] Hence find an expression for $\frac{d^2\sigma_H}{dydx}$ in terms of the PDFs.

END OF PAPER