UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 2 June, 2016 1:30 pm to 4:30 pm

PAPER 304

ADVANCED QUANTUM FIELD THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Write down the momentum space Feynman rules for the d-dimensional Euclidean scalar theory with action

$$S[\phi] = \int \left(\frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 \right) d^d x \,,$$

where λ is a coupling constant. Draw all connected, 1-loop, one particle irreducible Feynman graphs with four external lines.

Show that in d-dimensional Euclidean momentum space

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + m^2)^n} = \frac{1}{(n-1)!} \frac{(m^2)^{d/2-n}}{(4\pi)^{d/2}} \Gamma(n - d/2) \,,$$

where $k^2 = k_\mu k^\mu$. Use this result to verify that

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + m^2)((p-k)^2 + m^2)} \sim \frac{1}{8\pi^2 \epsilon} \qquad \text{as } \epsilon = 4 - d \to 0 \,.$$

Hence or otherwise compute the β-function for the quartic coupling in $S[\phi]$ to 1-loop accuracy in the $\overline{\text{MS}}$ scheme, and comment on its sign.

[You may assume that $\int d^d k \, e^{-ak^2} = (\pi/a)^{d/2}$ for $a > 0$, and that the Γ function $\Gamma(z) = \int_0^\infty ds \, s^{z-1} \, e^{-s}$ obeys $\Gamma(1) = 1$ and $\Gamma(1+z) = z\Gamma(z)$.]

2

Classical Yang-Mills theory has action

$$S[A] = \frac{1}{4g^2} \int F^a_{\mu\nu} F^{a\mu\nu} \, d^4x \,,$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a_{bc} A^b_\mu A^c_\nu$$

is the fieldstrength tensor for the gauge field A^a_μ. Obtain the Yang-Mills equations and the Bianchi identity obeyed by $F^a_{\mu\nu}$.

Without detailed calculation, briefly explain why these equations give a poor description of the phenomenology of quantum Yang-Mills theory at low energies.

Write down the additional terms that must be added to the above action if we wish to quantize Yang-Mills theory in some gauge $f^a[A] = 0$. State clearly the quantum numbers of any fields you introduce, and give a brief explanation of their role. What is the significance of choosing $f^a[A]$ to be a *local* functional?

In the case $f^a[A] = n^\mu A^a_\mu$ where n^μ is a fixed, constant vector, obtain the Feynman rules for the additional fields, and the (tree–level) momentum space propagator for the gauge field. Briefly explain why the resulting perturbation theory is Lorentz invariant.

3

Consider the action

$$S(z,\bar{z},\theta_i,\bar{\theta}_i) = |W'(z)|^2 - W''(z)\,\theta_1\theta_2 - \overline{W''(z)}\,\bar{\theta}_1\bar{\theta}_2$$

for a $d = 0$ theory, where $W(z)$ is a polynomial in the bosonic variable z and $W'(z) = \partial W/\partial z$. The variables θ_1, θ_2, $\bar{\theta}_1$ and $\bar{\theta}_2$ are Grassmann–valued.

Show that the Grassmann vector field

$$V_1 = \theta_1 \frac{\partial}{\partial z} - \overline{W'(z)} \frac{\partial}{\partial \theta_2}$$

obeys $\{V_1, V_1\} = 0$ and $V_1(S) = 0$. Find three further, independent Grassmann vector fields that also leave the action invariant.

Evaluate:

i) the partition function $Z = \int dz\, d\bar{z}\, d^2\theta\, d^2\bar{\theta}\, e^{-S(z,\bar{z},\theta_i,\bar{\theta}_i)}$,

ii) the (unnormalized) expectation value $\langle W'(z)g(z)\rangle$ where $g(z)$ is a polynomial, and

iii) the (unnormalized) expectation value $\langle f(z)\rangle$, where $f(z)$ is a further polynomial. [*Hint: Consider the effect of rescaling* $W(z)$.]

4

Consider the behaviour of a theory defined with UV cutoff Λ_0 as it flows down to a scale $\Lambda < \Lambda_0$. Write down the Callan–Symanzik equation describing the evolution of a correlation function $\langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2) \cdots \mathcal{O}_n(x_n) \rangle$ of composite local operators $\mathcal{O}_i(x_i)$, explaining the meaning of each of the terms.

A certain massless, four-dimensional theory has a single dimensionless coupling $g(\Lambda)$ at scales $\Lambda < \Lambda_0$. It also has a conserved current J_μ. The two–point function G_0 of these currents is defined by

$$(k_\mu k_\nu - \delta_{\mu\nu} k^2)\, G_0(k^2, g(\Lambda), \Lambda, \Lambda_0) = \int d^4x\, e^{-ik\cdot x}\, \langle J_\mu(x)\, J_\nu(0) \rangle \,.$$

What is the classical mass dimension of G_0?

G_0 diverges logarithmically as the cut–off $\Lambda_0 \to \infty$. At one–loop, a finite, renormalized two–point function $G_{\rm ren}(k^2, g(\Lambda), \Lambda)$ may be defined by

$$G_{\rm ren}(k^2, g(\Lambda), \Lambda) = \lim_{\Lambda_0 \to \infty} \left[G_0(k^2, g(\Lambda), \Lambda, \Lambda_0) \;-\; a \ln\left(\frac{\Lambda_0}{\Lambda}\right) \right] ,$$

where a is a constant. Write down the Callan–Symanzik equation obeyed by $G_{\rm ren}(k^2, g(\Lambda), \Lambda)$ assuming that the current $J_\mu(x)$ has vanishing anomalous dimensions.

Determine the leading behaviour of $G_{\rm ren}$ when $k^2 \gg \Lambda^2$ in the case that the theory is asymptotically free.

END OF PAPER