UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 30 May, 2016 1:30 pm to 3:30 pm

PAPER 303

STATISTICAL FIELD THEORY

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet

Treasury Tag

Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

[*For full credit attempt both parts of this question.*]

(a) Consider the Ising model on a square D-dimensional lattice which has N sites. The Hamiltonian is given by

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i$$

where i and j are site indices and the first sum is over nearest-neighbour pairs of sites. $J > 0$ is the interaction energy and h represents an external magnetic field (in energy units). The spin degrees-of-freedom $\sigma_i = \pm 1$. You may assume periodic boundary conditions and that each site has exactly $q = 2D$ nearest neighbours.

Making the mean-field approximation, find a simple, analytic expression for the partition function Z (i.e. explicitly evaluate all summations).

Denoting the magnetization-per-site by m, write down the mean-field free energy function $A(h, m)$ and expand about small m to obtain a power series including terms up to order m^4. In the $h = 0$ case, at what temperature is there a phase transition? What is the order of the phase transition?

Obtain a self-consistent expression for m in two ways: (i) as an expectation value; (ii) from the mean-field free energy.

Determine the (mean-field) critical exponents β, γ, and δ which parametrize the singular behaviour of the magnetization m and the magnetic susceptibility χ as follows

$$\begin{aligned} m &\sim (T_c - T)^\beta \quad \text{for } T < T_c \text{ and } h = 0 \\ \chi &\sim |T - T_c|^{-\gamma} \quad \text{for } h = 0 \\ m &\sim h^{1/\delta} \quad \text{for } T = T_c \,. \end{aligned}$$

(b) Consider instead a system which has free energy of the form

$$A = \frac{1}{2} A_2 m^2 + \frac{1}{3} A_3 m^3 + \frac{1}{4} A_4 m^4$$

with $A_3 \neq 0$ and $A_4 > 0$. Can this system possess a second-order phase transition? Can it possess a first-order phase transition? Justify your answers by finding relations between the coefficients A_k ($k = 2, 3, 4$) which determine whether the state is ordered or disordered.

2

Consider the $D = 2$ Ising model on a square lattice with lattice spacing a, and with nearest-neighbour (i.e. distance a apart) and next-to-nearest-neighbour (i.e. distance $a\sqrt{2}$ apart) interactions, respectively corresponding to the first and second terms in the Hamiltonian

$$H = -K \sum_{\langle ij \rangle} \sigma_i \sigma_j - L \sum_{\langle\langle k\ell \rangle\rangle} \sigma_k \sigma_\ell \,.$$

Perform a thinning of degrees-of-freedom by summing over the spins, "decimating" the spins, on every second site in a "checker-board" fashion – for example, decimate spins with (x, y) coordinates $(2m, 2n+1)$ and $(2m+1, 2n)$, with m and n integers. Rescale the lattice by a factor $b = \sqrt{2}$ to recover the original lattice spacing. Calculate the interactions on the blocked lattice keeping only terms up to $O(K^2)$ and $O(L)$. Show that there are only two such interactions to this order and that they are the same operators as the original ones with new couplings K' and L'.

Using the equations defining $K'(K, L)$ and $L'(K, L)$, find the critical points of the renormalization group transformation and identify the nontrivial fixed point. Linearizing about this point, find a value for the critical exponent ν.

3

[For full credit, attempt both parts of the question.]

The Landau-Ginzburg Hamiltonian for a scalar field in D-dimensions, $\phi(x)$, is given by

$$H[\phi, h] = \int d^D x \left[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2} r_0 \phi^2 + \frac{1}{4!} u_0 \phi^4 - h(x)\phi(x) \right]$$

and the partition function (or generating functional) is given by

$$Z[h] = \int \mathcal{D}\phi \, \exp\left(-H[\phi]\right) \, .$$

(a) Working within the Landau approximation, give expressions for the Helmholtz and Gibbs free energies, $F[h]$ and $\Gamma[m]$, respectively.

Show that the connected Green's function $G(x, y) = \langle \phi(y)\phi(x) \rangle_c$ satisfies the differential equation

$$K(x)G(x, y) = \delta(x - y)$$

where $\delta(x - y)$ is the Dirac δ-function and you should determine the differential operator $K(x)$ explicitly.

Let $\tilde{G}(q) = \int d^D x \, G(x) \, e^{iq \cdot x}$ and show that

$$\tilde{G}(q) = \frac{1}{q^2 + \xi^{-2}}$$

where you should find ξ and infer a value for the critical exponent ν.

(b) Letting $u_0 = 0$ but going beyond the Landau approximation, describe how r_0 and h behave under a renormalization group transformation which "integrates out" or "thins out" short wavelength Fourier modes. Here you may let h be constant in x.

END OF PAPER