UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 27 May, 2016 9:00 am to 12:00 pm

PAPER 302

SYMMETRIES, FIELDS AND PARTICLES

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 State the definition of a Lie algebra.

Let G be a matrix Lie group. By considering curves in G, show that the tangent space to G at the identity element, together with an appropriate bracket, defines a Lie algebra $\mathcal{L}(G)$.

In the case where G is $U(N)$, the group of $N \times N$ unitary matrices, determine the conditions obeyed by elements of the corresponding Lie algebra $\mathfrak{g} = \mathcal{L}(G)$ and find its dimension. Let $E^{(i,j)}$ be the $N \times N$ matrix with elements, $E^{(i,j)}_{\alpha\beta} = \delta_{\alpha\, i}\delta_{j\, \beta}$ where the indices α, β and labels i, j run from 1 to N. Write down a complete set of generators for $\mathfrak{g}$ consisting of appropriate linear combinations of the matrices $E^{(i,j)}$.

Now let $N = 2n$ where n is a positive integer. Consider the subset H of $U(2n)$ consisting of matrices $M \in U(2n)$ obeying,

$$M\mathbb{J}M^T = \mathbb{J} \qquad\qquad \mathbb{J} = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}$$

Here $\mathbb{I}_n$ denotes the $n \times n$ unit matrix. Show that H is a subgroup of $U(2n)$. Assuming it is a Lie subgroup, consider the corresponding Lie algebra $\mathfrak{h} = \mathcal{L}(H)$. Writing a general $2n \times 2n$ matrix as,

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

determine the constraints on the $n \times n$ submatrices A, B, C and D which are obeyed by elements M of $\mathfrak{h}$. Hence determine the dimension of $\mathfrak{h}$. Find a complete set of generators for $\mathfrak{h}$ consisting of appropriate linear combinations of the $2n \times 2n$ matrices $E^{(i,j)}$, $E^{(i+n,j)}$, $E^{(i,j+n)}$ and $E^{(i+n,j+n)}$, for $i, j = 1, 2, \ldots, n$.

2 Explain what is meant by a representation R of a Lie algebra $\mathfrak{g}$. Show that every non-abelian Lie algebra has at least one non-trivial representation of dimension $D = \dim(\mathfrak{g})$.

Define the Killing form κ on a Lie algebra $\mathfrak{g}$ and prove that it obeys the invariance condition,

$$\kappa([Z,X],Y) + \kappa(X,[Z,Y]) = 0 \qquad \forall X, Y, Z \in \mathfrak{g}$$

[*For convenience, you can set any normalisation constant appearing in your definition of* κ *to one.*]

Evaluate κ explicitly for,

i) $\mathfrak{g} = \mathfrak{su}(2) = \mathcal{L}(SU(2))$ in a basis $\{T^a = -i\sigma^a/2\}$, where σ^a for $a = 1, 2, 3$ denote the Pauli matrices.

ii) $\mathfrak{g} = \mathfrak{iso}(2)$ which is the Lie algebra generated by elements $\{J, E_1, E_2\}$ whose brackets are,

$$[J, E_1] = -E_2 \qquad [J, E_2] = E_1 \qquad [E_1, E_2] = 0$$

What is meant by the terms *simple* and *semi-simple* when referring to a Lie algebra $\mathfrak{g}$? Prove that, if κ is non-degenerate, then $\mathfrak{g}$ is necessarily semi-simple. Prove that, if $\mathfrak{g}$ is simple, then κ is necessarily non-degenerate. [*Here you may assume that* κ *does not vanish identically.*] Comment briefly on how these results apply to the examples considered in i) and ii) above.

3 [*In this question you can use without proof any general results that you need from the theory of simple, complex, finite-dimensional Lie algebras and their representations provided they are clearly stated.*]

Define the root lattice and the weight lattice of a simple, complex finite-dimensional Lie algebra $\mathfrak{g}$ of rank r with simple roots $\alpha_{(i)}$, $i = 1, \ldots, r$. Define the fundamental weights of $\mathfrak{g}$.

Starting from its Cartan matrix,

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

determine the weight lattice of A_2 explicitly by expressing the fundamental weights in terms of the simple roots. Sketch the resulting lattice, clearly marking the lattice vectors corresponding to the fundamental weights and those corresponding to simple roots.

What is meant by the *Dynkin indices* of a finite dimensional, irreducible representation R_Λ of $\mathfrak{g}$ with highest weight Λ? State without proof an algorithm to find the weights of R_Λ up to undetermined multiplicities.

Let $R(\Lambda_1, \Lambda_2)$ denote the finite-dimensional irreducible A_2 representation with Dynkin labels (Λ_1, Λ_2). Determine the weights of $R(2,0)$ and of the tensor product $R(2,0) \otimes R(2,0)$.

Determine the Dynkin labels of the irreducible components in the tensor product $R(2,0) \otimes R(2,0)$ explaining your reasoning carefully. Determine the weights of each irreducible component, indicating which weights, if any, are degenerate. [*You may use without proof the formula,*

$$\dim R(\Lambda_1, \Lambda_2) = \frac{1}{2}(\Lambda_1 + 1)(\Lambda_2 + 1)(\Lambda_1 + \Lambda_2 + 2)$$

to test for the presence of degenerate weights in each representation you consider. You may also use the fact that those weights of an irreducible A_2 representation R which lie on the boundary of the polygonal region in the weight lattice corresponding to R are never degenerate.]

4 Write an essay on non-abelian gauge invariance. Your account should include the construction of the Lagrangian density for Yang-Mills theory with an arbitrary compact, simple* gauge group G of finite dimension and also for a theory containing gauge fields coupled to scalar fields transforming in an arbitrary finite-dimensional, irreducible representation of G. You may restrict your discussion to infinitessimal gauge transformations.

[* *A simple Lie group G is such that the corresponding Lie algebra $\mathcal{L}(G)$ is simple.*]

END OF PAPER